MATH 422-Introduction to Topology

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Applications to Vector Spaces

In this chapter, we will briefly recall some basic facts about vector spaces, investigate various useful ways to define topologies on vector spaces, discuss the main differences between finite-dimensional vector spaces and infinite-dimensional vector spaces, and see some of the foundational theorems for investigating vector spaces with topologies.

The title of this chapter is "Applications to Vector Spaces" as we are going to be introducing an algebraic structure to our topological spaces. Once we do this, we are venturing away from a purely topological study and towards areas of mathematics such as Linear Algebra and Functional Analysis. Nevertheless, this is a course in topology, so we will mainly focus on the topological properties of these spaces.

The purpose of this chapter is to illustrate the necessity for a thorough understanding of topology in other areas of mathematics, in this case, a study of vector spaces. In this chapter, we will see that many of our important topological theorems, such as The Baire Category Theorem and Tychonoff's Theorem, are essential to answer some of our most fundamental questions about vector spaces. This chapter is by no means a thorough investigation into vector spaces endowed with a topology as this is an incredibly large field of study.

8.1 Vector Spaces

Most, if not all, of the material in this section should be familiar to the reader who has taken an introductory Linear Algebra course.

Definition 8.1.1. A **vector space** over a field $\mathbb{K}$ (the set $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$) is a set X, a mapping $+ : X \times X \to X$ called "addition" and a mapping $\cdot : \mathbb{K} \times X \to X$ called "scalar multiplication", where we denote $+(x, y) = x + y$ and $\cdot(\alpha, x) = \alpha x$, such that

(i) there exists $0 \in X$ such that $0 + x = x$, for all $x \in X$,

(ii) for all $x, y \in X$, we have $x + y = y + x$,

(iii) for all $x, y, z \in X$, we have $x + (y + z) = (x + y) + z$,

(iv) for all $x \in X$, there exists $-x \in X$ such that $x + (-x) = 0$,

(v) for all $\alpha, \beta \in \mathbb{K}$ and $x \in X$, we have $\alpha(\beta x) = (\alpha\beta)x$,

(vi) for all $\alpha, \beta \in \mathbb{K}$ and $x \in X$, we have $(\alpha + \beta)x = \alpha x + \beta x$,

(vii) for all $\alpha \in \mathbb{K}$ and $x, y \in X$, we have $\alpha(x + y) = \alpha x + \alpha y$, and

(viii) $1x = x$, for all $x \in X$.

Some immediate consequences of the above properties are that $0x = 0$ and $(-1)x = -x$. Also, the general convention is that $x - y$ is defined to be $x + (-y)$.

If X is a vector space over the field $\mathbb{R}$, then we call X a real vector space whereas, if X is a vector space over the field $\mathbb{C}$, then we call X a complex vector space. If we make a statement about a vector space X over the field $\mathbb{K}$, then the statement is true for $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$.

Example 8.1.2. (i) The real vector space that we are most familiar with from Linear Algebra would be $\mathbb{R}^n$, for some $n \in \mathbb{Z}_+$, where addition and scalar multiplication are defined component-wise. That is,

$$\begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \cdot \\ \cdot \\ x_n + y_n \end{bmatrix} \quad \text{and} \quad \alpha \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \cdot \\ \cdot \\ \alpha x_n \end{bmatrix}.$$

(ii) The most familiar complex vector space is $\mathbb{C}^n$, for some $n \in \mathbb{Z}_+$, where addition and scalar multiplication are define component-wise.

(iii) The set $\mathbb{R}^{\mathbb{R}}$ forms a real vector space where we define addition by $(f+g)(x) = f(x)+g(x)$ and scalar multiplication by $(\alpha f)(x) = \alpha f(x)$.

(iv) The set of all real-valued sequences $\mathbb{R}^{\aleph_0}$ is a real vector space where $(x_n)_{n=1}^{\infty} + (y_n)_{n=1}^{\infty} = (x_n + y_n)_{n=1}^{\infty}$ and $\alpha(x_n)_{n=1}^{\infty} = (\alpha x_n)_{n=1}^{\infty}$.

The observant reader has perhaps noticed that all of the examples above are particular cases of a more general example. Given our field $\mathbb{K}$, which is either $\mathbb{R}$ or $\mathbb{C}$, the product space $\mathbb{K}^I$ is a vector space where we define the addition of two elements $f : I \to \mathbb{K}$ and $g : I \to \mathbb{K}$ by $(f+g)(i) = f(i)+g(i)$ and scalar multiplication by $(\alpha f)(i) = \alpha f(i)$. In example (ii), $\mathbb{K} = \mathbb{C}$ and $I = \mathbb{Z}_n$, while in examples (i), (iii), and (iv), $\mathbb{K} = \mathbb{R}$, while $I = \mathbb{Z}_n$ in (i), $I = \mathbb{R}$ in (iii), and $I = \mathbb{Z}_+$ in (iv).

Definition 8.1.3. Let X be a vector space and let V be a subset of X. We say V is a **subspace** of X, if

(i) for all $x, y \in V$, we have that $x + y \in V$, and

(ii) for all $\alpha \in \mathbb{K}$ and $x \in V$, we have that $\alpha x \in V$.

Note that if X is a vector space and V is a subspace of X then V is a vector space itself. It inherits the commutative, distributive, and associative properties from X while property (ii) implies that if $x \in V$, then $-x = (-1)x \in V$ and so property (i) implies $0 = x + (-x) \in V$.

Example 8.1.4. (i) Recall the sets c_{00}, c_0, and c from Chapter 1. The set c_{00} is the set of all real-valued sequences with finitely many nonzero coordinates, the set c_0 is the set of all real-valued sequences which converge to zero, and the set c is the set of all real-valued sequence which converge. All three sets are examples of subspaces of $\mathbb{R}^{\aleph_0}$. Further, we have that $c_{00} \subseteq c_0 \subseteq c$ so c_{00} and c_0 are also subspaces of c while c_{00} is a subspace of c_0.

(ii) From Corollary **??** we know that if $f, g \in C(\mathbb{R})$ (recall that $C(\mathbb{R})$ is the set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$) then $f + g \in C(\mathbb{R})$ and $\alpha f \in C(\mathbb{R})$ thus, $C(\mathbb{R})$ is a subspace of $\mathbb{R}^{\mathbb{R}}$.

(iii) Let $\mathfrak{B}(\mathbb{R})$ be the set of all bounded functions $f : \mathbb{R} \to \mathbb{R}$. Then $\mathfrak{B}(\mathbb{R})$ is a subspace of $\mathbb{R}^{\mathbb{R}}$.

(iv) Recall that $C([a, b])$ is the set of all continuous functions $f : [a, b] \to \mathbb{R}$ and let $\mathfrak{B}([a, b])$ be the set of all bounded functions $f : [a, b] \to \mathbb{R}$. Then $C([a, b])$ and $\mathfrak{B}([a, b])$ are subspaces of $\mathbb{R}^{[a,b]}$. Let $f \in C([a, b])$. Since $[a, b]$ is compact and f is continuous, the set $f([a, b])$ is compact. Then, by The Heine-Borel Theorem, $f([a, b])$ is bounded and so $f \in \mathfrak{B}([a, b])$. Thus, $C([a, b])$ is a subspace of $\mathfrak{B}([a, b])$.

(v) Let $P([a, b])$ denote the set of all polynomials from $[a, b]$ into $\mathbb{R}$. Then $P([a, b])$ is a subspace of $C([a, b])$. Further, if $P(\mathbb{R})$ is the set of all polynomials from $\mathbb{R}$ into $\mathbb{R}$, then $P(\mathbb{R})$ is a subspace of $C(\mathbb{R})$. Further, if $P_n([a, b])$ denotes the set of all polynomials of degree less than or equal to n defined on $[a, b]$, then $P_n([a, b])$ is a subspace of $P([a, b])$ while $P_n(\mathbb{R})$ (the set of all polynomials of degree less than or equal to n defined on $\mathbb{R}$), then $P_n(\mathbb{R})$ is a subspace of $P(\mathbb{R})$.

Definition 8.1.5. Let X be a vector space over the field $\mathbb{K}$ and let $B \subseteq X$. We say the set B is **linearly independent** if for every $n \in \mathbb{Z}_+$ and every $x_1, x_2, \ldots, x_n \in B$ the statement

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0, \quad \text{for some } \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{K}$$

implies

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

Definition 8.1.6. If X is a vector space and $x_1, x_2, \ldots, x_n \in X$, for some $n \in \mathbb{Z}_+$, then a **linear combination** of $x_1, x_2, \ldots, x_n$ is a vector of the form $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$ for some $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{K}$.

Definition 8.1.7. If X is a vector space and V is a subset of X, then the **linear subspace generated by** V or the **span of** V is the set of all linear combinations of elements of V and is denoted by $Span(V)$.

Exercise 8.1.8. Let X be a vector space over $\mathbb{K}$ and let $V \subseteq X$. Prove that $Span(V)$ is a subspace of X.

Definition 8.1.9. A subset B of a vector space X is called a **Hamel basis** of X if B is linearly independent and $Span(B) = X$.

Example 8.1.10. (i) Consider the vector space $\mathbb{R}^n$, for some $n \in \mathbb{Z}_+$ and let $B = \{e_1, e_2, \ldots, e_n\}$, where

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{bmatrix}, \ldots, e_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdot \\ \cdot \\ 1 \end{bmatrix}.$$

Then B is a Hamel basis for $\mathbb{R}^n$.

(ii) Consider the vector space $P(\mathbb{R})$ and the set $B = \{1, x, x^2, x^3, \ldots\}$. Then B is a Hamel basis for $P(\mathbb{R})$.

(iii) Consider the vector space $\mathbb{R}^{\aleph_0}$. Then $\mathbb{R}^{\aleph_0}$ does not have a Hamel basis nor does its subspaces c_0 or c. If we consider the subspace c_{00}, then $B = \{e_1, e_2, e_3, \ldots\}$ is a Hamel basis for c_{00} where $e_1 = (1, 0, 0, 0, \ldots)$, $e_2 = (0, 1, 0, 0, \ldots)$, $e_3 = (0, 0, 1, 0, \ldots)$, etc.

(iv) The vector spaces $C([a, b])$, $\mathfrak{B}([a, b])$, $C(\mathbb{R})$, $\mathfrak{B}(\mathbb{R})$, and $\mathbb{R}^{\mathbb{R}}$ do not have Hamel bases.

It should be stressed that linear combinations of vectors are defined to be *finite* sums of scalar multiples of vectors. Thus, for a set B to span a vector space X, every element of X needs to be expressable as a *finite* sum of scalar multiples of elements of B. The reader might wonder why we don't allow for infinite sums but, remember, an infinite sum is defined as the limit of its sequence of partial sums. We cannot discuss limits of sequences without any topological structure. If we want to discuss a different kind of basis (no longer a Hamel basis) which allows for infinite sums, then we would need to introduce a topology to our vector space.

Definition 8.1.11. Let X be a vector space and suppose B is a Hamel basis for X. If $|B| = n$, for some $n \in \mathbb{Z}_+$, then we say the vector space X is **finite dimensional** and say X is an n-dimensional vector space. We write $dim(X) = n$. Otherwise, we say the vector space X is **infinite dimensional**.

The next exercise checks that our above definition makes sense.

Exercise 8.1.12. Let X be a finite dimensional vector space and suppose B and C are Hamel bases for X. Prove $|B| = |C|$.

Definition 8.1.13. Let X and Y be vector spaces over the same field $\mathbb{K}$. We say a function $T : X \to Y$ is a **linear operator** (or **linear transformation**) if

(i) $T(x + y) = T(x) + T(y)$ for all $x, y \in X$, and

(ii) $T(\alpha x) = \alpha T(x)$ for all $\alpha \in \mathbb{K}$ and all $x \in X$.

Note that if T is a linear operator, then $T(0) = 0$ from the properties above. Also,for a linear operator T, we often write $T(x) = Tx$. If X and Y are vector spaces, then we denote the set of all linear operators $T : X \to Y$ by $L(X, Y)$. If $X = Y$ then we write $L(X)$ rather than $L(X, X)$. In Linear Algebra, we learn that when $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, then $L(X, Y)$ is precisely the set of all $m \times n$ matrices with real-valued entries. We will denote the set of all $m \times n$ matrices with real-valued entries by

$M_{m\times n}(\mathbb{R})$. Thus, we have $L(\mathbb{R}^n, \mathbb{R}^m) = M_{m\times n}(\mathbb{R})$. We also learn in linear algebra that $M_{m\times n}(\mathbb{R})$ is a vector space with matrix addition and scalar multiplication. The same is true for $L(X, Y)$ for any vector spaces X and Y as the reader is asked to prove in the next exercise.

Exercise 8.1.14. Let X and Y be vector spaces over the same field $\mathbb{K}$. Prove $L(X, Y)$ is a vector space over the field $\mathbb{K}$ where we define $(L + M)(x) = Lx + Mx$ for all $L, M \in L(X, Y)$ and $x \in X$ and $(\alpha L)(x) = \alpha Lx$ for all $\alpha \in \mathbb{K}$ and $x \in X$. *Note that $L(X, Y) \subseteq Y^X$ where we defined vector addition and scalar multiplication component-wise so it suffices to prove that $L(X, Y)$ is a subspace of Y^X.*

For those who have taken an abstract algebra course, a homomorphism between two algebraic structures of the same type (groups, rings, vector spaces, etc.) is a function from one structure to the other which respects the operations. Thus, the set of all linear operators $L(X, Y)$ is precisely the set of all vector space homomorphisms from X to Y. More specifically, $M_{m\times n}(\mathbb{R})$ is the set of all vector space homomorphisms from $\mathbb{R}^n$ to $\mathbb{R}^m$.

An important property for linear operators is whether or not they are continuous. Currently, we have no way to discuss the continuity of linear operators though because, once again, continuity requires a topological structure. This leads us to the topic of the next section.

8.2 Vector Topologies

Definition 8.2.1. Let X be a vector space and let τ be a topology on X such that the addition map $+ : (X \times X, \sigma) \to (X, \tau)$ is continuous, where σ is the product topology on $X \times X$, and the scalar multiplication map $\cdot : (\mathbb{K} \times X, \gamma) \to (X, \tau)$ is continuous, where γ is the product topology on $\mathbb{K} \times X$, where $\mathbb{K}$ has the usual topology. Then we say τ is a **vector topology** and refer to (X, τ) as a **topological vector space** or **linear topological space**.

Thus, a topological vector space is a vector space endowed with a topology which makes the operations continuous. That is, given nets $(x_\lambda)_{\lambda\in\Lambda}, (y_\lambda)_{\lambda\in\Lambda}$ in X and a net $(\alpha_\lambda)_{\lambda\in\Lambda}$ in $\mathbb{K}$ we have that, if $x_\lambda \to x$, $y_\lambda \to y$, and $\alpha_\lambda \to \alpha$, for some $x, y \in X$ and $\alpha \in \mathbb{K}$, then $(x_\lambda + y_\lambda) \to (x + y)$ and $\alpha_\lambda x_\lambda \to \alpha x$.

The most prevalent types of topologies given to vector spaces are weak topologies generated by sets of functions and topologies generated by norms. We will start by looking at topologies generated by norms.

Definition 8.2.2. Let X be a vector space over the field $\mathbb{K}$. A **norm** on X is a map $\|\cdot\| : X \to \mathbb{R}$, where we denote $\|\cdot\|(x)$ by $\|x\|$, such that

(i) $\|x\| \geq 0$, for all $x \in X$,

(ii) $\|x\| = 0$ if and only if $x = 0$,

(iii) $\|\alpha x\| = |\alpha|\|x\|$, for all $x \in X$ and $\alpha \in \mathbb{K}$, and

(iv) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in X$.

If $\|\cdot\|$ is a norm on X, then we call $(X, \|\cdot\|)$ a **normed vector space**.

Note that in property (iii) above, if $\mathbb{K} = \mathbb{R}$, then $|\alpha|$ is the absolute value of α while, if $\mathbb{K} = \mathbb{C}$, then $|\alpha|$ is the modulus of α. We refer to $\|x\|$ as *the norm of* x. We typically think of $\|x\|$ as the magnitude, or size, of the vector x.

Example 8.2.3. (i) We can define $\mathbb{R}$ to be a vector space over the field $\mathbb{R}$ and define the norm by $\|x\| = |x|$ for all $x \in \mathbb{R}$. It is easy to check properties (i) through (iii) in the definition of the norm. Property (iv) is simply the triangle inequality for the usual metric d on $\mathbb{R}$. That is,

$$|x+y| = d(x,-y) \leq d(x,0) + d(0,-y) = |x| + |y|.$$

For this reason, property (iv) of the definition of a norm is usually referred to as the triangle inequality for norms. Thus, $(\mathbb{R}, \|\cdot\|)$ is a normed space.

(ii) Let $n \in \mathbb{Z}_+$ and consider $\mathbb{K}^n$ to be a vector space over $\mathbb{K}$. Let d_2 be the usual metric on $\mathbb{K}^n$ and define $\|x\| = d_2(x,0)$. Then $\|\cdot\|$ is a norm on $\mathbb{K}^n$. Proving this fact is an exercise below. We typically denote the norm here by $\|\cdot\|_2$ and it is referred to as the ℓ_2-norm on $\mathbb{K}^n$ or the Euclidean norm on $\mathbb{K}^n$.

(iii) The set c of convergent sequences of real numbers is a vector space over $\mathbb{R}$. Let $(x_n)_{n=1}^{\infty} \in c$. Recall that convergent sequences are bounded. Thus, $\{|x_n| : n \in \mathbb{Z}_+\}$ is a set of real numbers which has an upper bound. Hence, it has a least upper bound. Define $\|(x_n)_{n=1}^{\infty}\| = \sup\{|x_n| : n \in \mathbb{Z}_+\}$. Then $\|\cdot\|$ is a norm on c. Proving this fact is an exercise below. We typically denote this norm by $\|\cdot\|_\infty$. Thus, $(c, \|\cdot\|_\infty)$ is a normed space.

(iv) The vector space $C([a,b])$ is a normed space if we define $\|f\| = \sup\{|f(x)| : x \in [a,b]\}$ and recall that continuous functions on compact sets must be bounded. Thus, the set $f([a,b])$ has a least upper bound. We typically denote this norm by $\|\cdot\|_\infty$ and refer to it as the supremum norm, or by $\|\cdot\|_{\mathcal{U}}$ where the $\mathcal{U}$ refers to the "uniform norm." For those who have taken an analysis course, a sequence of functions in $(C([a,b]), \|\cdot\|_\infty)$ converges to a function f precisely when the sequence converges *uniformly* to f. Hence, the use of the letter $\mathcal{U}$.

Exercise 8.2.4. (i) Prove $(c, \|\cdot\|_\infty)$ is a normed space.

(ii) Prove $(\mathbb{K}^n, \|\cdot\|_2)$ is a normed space. *Hint: You do not have to prove the triangle inequality from scratch. It is much easier to use the triangle inequality for the metric d_2 to prove the triangle inequality for the norm $\|\cdot\|_2$.*

If $(X, \|\cdot\|)$ is a normed space, then we can always define a metric d on X by $d(x, y) = \|x - y\|$ for all $x, y \in X$. If you recall in Chapter 2 when discussing the motivation for metric spaces, we defined them the way that we did because we did not always have a meaningful way to define substraction on our set or to define the magnitude of an element of the set. For normed spaces, we have both so we have come full circle and are able to define distance in this way. Since we can define a metric d on X, we can define the topology τ generated by d. Thus, (X, τ) becomes a topological vector space whose topology is generated by a metric. When discussing the normed space $(X, \|\cdot\|)$, it is standard to assume that we are considering it a topological vector space whose topology is generated by the metric d above. Thus, normed spaces are metric spaces and everything that we have learned in previous chapters about metric spaces applies to normed spaces. One important property about metric spaces, as we have already seen, is completeness.

Definition 8.2.5. A complete normed space $(X, \|\cdot\|)$ is called a **Banach space**.

Just to reiterate what we were discussing in the previous paragraph, when we say the normed space $(X, \|\cdot\|)$ is complete, we mean the metric space (X, d) is complete, where $d(x, y) = \|x - y\|$ for all $x, y \in X$. Whenever we discuss topological properties about a normed space $(X, \|\cdot\|)$ it is because the metric space (X, d) has those properties.

Example 8.2.6. (i) For any $n \in \mathbb{Z}_+$, the normed space $(\mathbb{R}^n, \|\cdot\|_2)$ is a Banach space. Let $(\overline{x}_k)_{k=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}^n$, where $\overline{x}_k = (x_{1,k}, x_{2,k}, \ldots, x_{n,k})$. Fix $i = 1, 2, \ldots, n$ and consider the sequence $(x_{i,k})_{k=1}^{\infty}$ in $\mathbb{R}$. For $\epsilon > 0$, pick $K \in \mathbb{Z}_+$ such that, for all $k_1, k_2 \geq K$, we have that $\|\overline{x}_{k_1} - \overline{x}_{k_2}\| < \epsilon$. Thus,

$$|x_{i,k_1} - x_{i,k_2}| \leq \left(\sum_{j=1}^{n} |x_{j,k_1} - x_{j,k_2}|^2 \right)^{1/2} < \epsilon$$

and so $(x_{i,k})_{k=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, there exists $y_i \in \mathbb{R}$ such that $x_{i,k} \xrightarrow{k\to\infty} y_i$. Let $\overline{y} = (y_1, y_2, \ldots, y_n)$. Let $\epsilon > 0$. Since there are only finitely many coordinates in $\mathbb{R}^n$, pick $K \in \mathbb{Z}_+$ large enough so that $|x_{i,k} - y_i| < \frac{\epsilon}{\sqrt{n}}$ for all $k \geq K$ and all $i = 1, 2, \dots, n$. Then,

$$\|\overline{x}_k - \overline{y}\| = \left(\sum_{i=1}^{n} |x_{i,k} - y_i|^2 \right)^{1/2} \leq \left(\sum_{i=1}^{n} \left(\frac{\epsilon}{\sqrt{n}} \right)^2 \right)^{1/2} = \epsilon.$$

Hence, $\overline{x}_k \to \overline{y}$. Thus, $(\mathbb{R}^n, d_2)$ is complete and so $(\mathbb{R}^n, \|\cdot\|_2)$ is a Banach space. A similar argument shows that $\mathbb{C}^n$ is also a Banach space over the field $\mathbb{C}$ when we define $\|\overline{z}\| = d(\overline{z}, 0)$, where d is the usual metric on $\mathbb{C}^n$.

(ii) The normed space $(c_{00}, \|\cdot\|_\infty)$ is not a Banach space. Note that c_{00} is a subspace of c so the norm $\|\cdot\|_\infty$ is well-defined on c_{00}. To see that c_{00} is not complete with this norm, for all $n \in \mathbb{Z}_+$, let $\overline{x}_n = (1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, 0, \dots)$. Then $(\overline{x})_{n=1}^{\infty}$ is a Cauchy sequence in c_{00}. Indeed, let $\epsilon > 0$. Pick $N \in \mathbb{Z}_+$ such that $N > \frac{1}{\epsilon}$. Then, for $n > m \geq N$, we have

$$\|\overline{x}_n - \overline{x}_m\| = \sup \left\{ \frac{1}{m+1}, \frac{1}{m+2}, \dots, \frac{1}{n} \right\} = \frac{1}{m+1} < \frac{1}{m} < \frac{1}{N} < \epsilon.$$

However, the sequence $(\overline{x}_n)_{n=1}^{\infty}$ does not converge to an element of c_{00}. As a sequence inside of c, we have that $\overline{x}_n \to \overline{x}$ where $\overline{x} = (\frac{1}{k})_{k=1}^{\infty}$ but $\overline{x} \notin c_{00}$.

(iii) The normed space $(C([a,b]), \|\cdot\|_\infty)$ is a Banach space although this fact requires some introductory analysis to justify. For those who have taken analysis, since continuous functions on compact sets are uniformly continuous and sequences converge in $C([a,b])$ precisely when they converge uniformly, the limit of any convergent sequence would be continuous on $[a,b]$ and thus an element of $C([a,b])$.

Definition 8.2.7. Let $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ be normed spaces over the same field and let $T \in L(X,Y)$. We say the linear operator T is **bounded** if the set $\{\|Tx\|_2 \,|\, x \in X \text{ and } \|x\|_1 \leq 1\}$ is bounded.

The reader is asked to prove a very useful inequality in the next exercise.

Exercise 8.2.8. Let $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ be normed spaces over the same field and let $T \in L(X, Y)$. If T is bounded, then there exists $C > 0$ such that, for all $x \in X$, we have $\|Tx\|_2 \leq C\|x\|_1$.

Theorem 8.2.9. *Let $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ be normed spaces over the same field and let $T \in L(X, Y)$. The linear operator T is continuous if and only if it is bounded.*

Proof. We will prove the forward direction by contrapositive so suppose $\{\|Tx\|_2 \,|\, x \in X \text{ and } \|x\|_1 \leq 1\}$ is not bounded. Then, for every $n \in \mathbb{Z}_+$, there exists $y_n \in \{\|Tx\|_2 \,|\, x \in X \text{ and } \|x\|_1 \leq 1\}$ such that $\|Ty_n\|_2 \geq n$. Note that $\|y_n\|_1 = 1$, for all $n \in \mathbb{Z}_+$. Let $x_n = \frac{1}{n} y_n$. Then,

$$\|x_n - 0\|_1 = \frac{1}{n}\|y_n\|_1 = \frac{1}{n} \to 0,$$

thus, $x_n \to 0$. But,

$$\|Tx_n - T0\|_2 = \|Tx_n\|_2 = \frac{1}{n}\|Ty_n\|_2 = 1$$

and so $Tx_n \nrightarrow T0$. Hence, T is not continuous.

For the other direction, suppose $\{\|Tx\|_2 \,|\, x \in X \text{ and } \|x\|_1 \leq 1\}$ is bounded. From Exercise 8.2.8, there exists $C > 0$ such that $\|Tx\|_2 \leq C\|x\|_1$, for all $x \in X$. Let $(x_n)_{n=1}^{\infty}$ be a sequence in X such that $x_n \to x$, for some $x \in X$. Then,

$$\|Tx_n - Tx\|_2 \leq C\|x_n - x\|_1 \to 0$$

and so $Tx_n \to Tx$. Hence, T is continuous. □

Thus, for linear operators, the words "continuous" and "bounded" are synonymous. We tend to say "bounded" rather than "continuous" though. If $(X, \|\cdot\|_1)$ and

$(Y, \|\cdot\|_2)$ are normed spaces over the same field, we denote the set of all bounded linear operators $T \in L(X,Y)$ by $\mathcal{B}(X,Y)$. If $(X, \|\cdot\|_1) = (Y, \|\cdot\|_2)$, then we write $\mathcal{B}(X)$ instead of $\mathcal{B}(X,X)$.

Exercise 8.2.10. Let $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ be normed spaces over the same field. Prove that $\mathcal{B}(X,Y)$ is a vector space. *Hint: We already know that $L(X,Y)$ is a vector space and $\mathcal{B}(X,Y) \subseteq L(X,Y)$, so it suffices to show that $\mathcal{B}(X,Y)$ is a subspace of $L(X,Y)$.*

Notice that if $T \in \mathcal{B}(X,Y)$, then, by definition, $\{\|Tx\|_2 | \, x \in X \text{ and } \|x\| \leq 1\}$ is a bounded subset of $\mathbb{R}$. Thus, it has a least upper bound. Hence, we can define

$$\|T\| = \sup\{\|Tx\|_2 | \, x \in X \text{ and } \|x\| \leq 1\}.$$

With the norm $\|\cdot\|$, the set $\mathcal{B}(X,Y)$ is a normed space as the next proposition verifies.

Proposition 8.2.11. *Let $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ be normed spaces over the same field. For every $T \in \mathcal{B}(X,Y)$, define*

$$\|T\| = \sup\{\|Tx\|_2 | \, x \in X \text{ and } \|x\| \leq 1\}.$$

Then $\|\cdot\|$ is a norm on $\mathcal{B}(X,Y)$.

Proof. To verify property (i) of the definition of a norm, note that for any $x \in X$, we have $\|Tx\|_2 \geq 0$, thus $\|T\| \geq 0$.

For property (ii), if $\|T\| = 0$, then $\|Tx\|_2 = 0$ for all $x \in X$ such that $\|x\|_1 \leq 1$. Then, for any $x \in X \setminus \{0\}$ we have that $\|\frac{x}{\|x\|_1}\|_1 \leq 1$ so $\|\frac{Tx}{\|x\|_1}\|_2 = 0$. Hence, $\|Tx\|_2 = 0$ and so $Tx = 0$. Thus, we have that $Tx = 0$ for all $x \in X$ and so $T = 0$. The other direction is trivial since, if $T = 0$, then $\{\|Tx\|_2 \,|\, x \in X \text{ and } \|x\|_1 \leq 1\} = \{0\}$.

For property (iii), let $x \in X$ such that $\|x\|_1 \leq 1$ and let $\alpha \in \mathbb{K}$. Then,

$$\|\alpha Tx\|_2 \leq |\alpha|\|Tx\| \leq |\alpha|\|T\|$$

and so $|\alpha|\|T\|$ is an upper bound for $\{\|\alpha Tx\|_2 \,|\, x \in X \text{ and } \|x\|_1 \leq 1\}$, and so, $\|\alpha T\| \leq |\alpha|\|T\|$. For the reverse inequality, let *epsilon* > 0. Then there exists $x \in X$, where $\|x\|_1 \leq 1$ such that $\|T\| \leq \|Tx\|_2 - \epsilon(|\alpha|+1)^{-1}$. Thus,

$$|\alpha|\|T\| \leq \|\alpha Tx\|_2 - \epsilon \leq \|\alpha T\| - \epsilon.$$

Since this holds for all $\epsilon > 0$, we have that $|\alpha|\|T\| \leq \|\alpha T\|$ and therefore $|\alpha|\|T\| = \|\alpha T\|$.

For property (iv), let $T, S \in \mathcal{B}(X,Y)$ and let $x \in X$ such that $\|x\|_1 \leq 1$. Then,

$$\|(T+S)x\|_2 \leq \|Tx\|_2 + \|Sx\|_2 \leq \|T\| + \|S\|.$$

Thus, $\|T\| + \|S\|$ is an upper bound for the set $\{\|(T+S)x\|_2 \,|\, x \in X \text{ and } \|x\|_1 \leq 1\}$, and so $\|T+S\| \leq \|T\| + \|S\|$.

Hence, we have shown that $\|\cdot\|$ is a norm on $\mathcal{B}(X,Y)$. □

The norm defined on $\mathcal{B}(X,Y)$ in the previous proposition is often referred to as the **operator norm** as there are other ways to define norms on sets of bounded linear operators. Notice that if $T \in \mathcal{B}(X,Y)$, then, for any $x \in X \setminus \{0\}$, we have that $\|T(\frac{x}{\|x\|_1})\|_2 \leq \|T\|$. Thus, for all $x \in X$, we have the inequality $\|Tx\|_2 \leq \|T\|\|x\|_1$.

Obviously, we would like to know when $\mathcal{B}(X,Y)$ is a Banach space. The next theorem provides us with an answer to this question but we first need an exercise.

Exercise 8.2.12. Let $(X, \|\cdot\|)$ be a normed space.

(i) Prove that for all $x, y \in X$, we have $|\|x\| - \|y\|| \leq \|x - y\|$.

(ii) Use the above inequality to prove that if $(x_n)_{n=1}^{\infty}$ is a sequence in X such that $x_n \to x$, for some $x \in X$, then $\|x_n\| \to \|x\|$.

Theorem 8.2.13. *Let $(X, \|\cdot\|_1)$ be a normed space over the field $\mathbb{K}$ and let $(Y, \|\cdot\|_2)$ be a Banach space over the field $\mathbb{K}$. Then $\mathcal{B}(X, Y)$ is a Banach space.*

Proof. Let $(T_n)_{n=1}^{\infty}$ be a Cauchy sequence in $\mathcal{B}(X, Y)$.

Claim: For all $x \in X$, we have that $(T_n x)_{n=1}^{\infty}$ is a Cauchy sequence in Y.

Let $x \in X \setminus \{0\}$ and let $\epsilon > 0$. Since $(T_n)_{n=1}^{\infty}$ is Cauchy, there exists $N \in \mathbb{Z}_+$ such that, for all $n, m \geq N$, we have that $\|T_n - T_m\| < \frac{\epsilon}{\|x\|_1}$. Thus,

$$\|T_n x - T_m x\|_2 \leq \|T_n - T_m\| \|x\|_1 < \epsilon$$

and so $(T_n x)_{n=1}^{\infty}$ is a Cauchy sequence in Y.

Since Y is complete, there exists $y_x \in Y$ such that $T_n x \to y_x$. Define $T : X \to Y$ by $Tx = y_x$. Thus, we have our candidate for the limit of the sequence $(T_n)_{n=1}^{\infty}$. It is left to show that T is linear, bounded, and that $T_n \to T$.

First, we show T is linear. Let $x_1, x_2 \in X$. Then,

$$T(x_1+x_2) = \lim_{n\to\infty} T_n(x_1+x_2) = \lim_{n\to\infty} (T_n x_1+T_n x_2) = \lim_{n\to\infty} T_n x_1+ \lim_{n\to\infty} T_n x_2 = Tx_1+Tx_2.$$

Similarly, for $\alpha \in \mathbb{K}$ and $x \in X$, we have

$$T(\alpha x) = \lim_{n\to\infty} T_n(\alpha x) = \alpha \lim_{n\to\infty} T_n x = \alpha T x.$$

Hence, T is linear.

Now, to show T is bounded, pick $N \in \mathbb{Z}_+$ such that, for all $n, m \geq N$, we have that $\|T_n - T_m\| \leq 1$. Then, for $x \in X$ such that $\|x\| \leq 1$, we have from Exercise 8.2.13

that $\|Tx\|_2 = \lim_{n\to\infty} \|T_n x\|_2$. Thus,

$$\begin{aligned}\|Tx\|_2 &= \lim_{n\to\infty} \|T_n x\|_2 \\ &\leq \lim_{n\to\infty} \|T_n x - T_N x\|_2 + \|T_N x\|_2 \\ &\leq \lim_{n\to\infty} \|T_n - T_N\| \|x\|_1 + \|T_N x\|_2 \\ &\leq \|x\|_1 + \|T_N\| \\ &\leq 1 + \|T_N\|.\end{aligned}$$

Thus, T is bounded.

Lastly, to show $T_n \to T$, let $\epsilon > 0$ and pick $N \in \mathbb{Z}_+$ such that, for all $n, m \geq N$, we have that $\|T_n - T_m\| < \frac{\epsilon}{2}$. Then, for $x \in X$ such that $\|x\|_1 \leq 1$ and $n \geq N$, we have

$$\|(T_n - T)x\|_2 = \lim_{m\to\infty} \|(T_n - T_m)x\|_2 \leq \|T_n - T_m\| \|x\|_1 \leq \frac{\epsilon}{2}\|x\|_1 \leq \frac{\epsilon}{2} < \epsilon.$$

Hence, $T_n \to T$ and so $\mathcal{B}(X, Y)$ is complete. □

Thus, $\mathcal{B}(X, Y)$ is a Banach space if its codomain is a Banach space. Of particular interest is when the codomain is $\mathbb{R}$.

Definition 8.2.14. Let $(X, \|\cdot\|)$ be a normed space over the field $\mathbb{K}$. A **linear functional** is a linear operator $f : X \to \mathbb{K}$. The set of all *bounded* linear functionals from X into its scalar field $\mathbb{K}$ is denoted by X^* and is called the **dual** of the normed space X.

Exercise 8.2.15. Let $(X, \|\cdot\|)$ be a normed space and let f be a linear functional on X. The functional f is bounded if and only if $f^{-1}(\{0\})$ is closed.

An immediate consequence of Theorem 8.2.13 and the above definition is the following corollary.

Corollary 8.2.16. *If $(X, \|\cdot\|)$ is a normed space, then X^* is a Banach space.*

Proof. The set $X^* = \mathcal{B}(X, \mathbb{K})$. Since $\mathbb{K}$ can be thought of as a Banach space over the field $\mathbb{K}$, Theorem 8.2.13 finishes the proof. □

Given a normed space $(X, \|\cdot\|_1)$, we can now discuss its dual Banach space $(X^*, \|\cdot\|)$. Thus, we can discuss the dual of $(X^*, \|\cdot\|)$, which we denote by X^{**} and refer to it as the **bidual** of the normed space $(X, \|\cdot\|_1)$. Note that the bidual is also a Banach space. We can continue in this fashion to define X^{***}, X^{****}, etc. but we almost never have a need to define more than the dual and bidual.

There are many motivations for studying the dual and bidual of a normed space and we will see some of them later in this chapter. One motivation is that it gives us a convenient way to describe the completion of a normed space. In order to do this, we first need another important Theorem called The Hahn-Banach Extension Theorem.

8.3 Hahn-Banach Extension Theorem

The Hahn-Banach Extension Theorem gives us a convenient way to extend a functional defined on a subspace of a vector space to the entire vector space. There are more general statements of the Hahn-Banach Extension Theorem but the version we will look at in this section will suffice for our purposes. Before seeing the theorem, we first need a lemma.

Lemma 8.3.1. *Let $(X, \|\cdot\|)$ be a real vector space and suppose there exists a map $p : X \to \mathbb{R}$ such that*

(i) $p(x + y) \le p(x) + p(y)$, for all $x, y \in X$, and

(ii) $p(tx) = tp(x)$, for all $t \in [0, \infty)$.

Further, suppose $\phi_0 : M \to \mathbb{R}$ is a linear map and $\phi_0(x) \le p(x)$ for all $x \in M$. Then, *there exists a linear map $\phi : X \to \mathbb{R}$ such that $\phi|_M = \phi_0$ and $\phi(x) \le p(x)$, for all $x \in X$.*

Proof. Let $\mathcal{P}$ be the set of all ordered pairs (Y, γ), where Y is a subspace of X which contains M and $\gamma : Y \to \mathbb{R}$ is a linear map such that $\gamma|_M = \phi_0$ and $\gamma(x) \le p(x)$ for all $x \in Y$. Define the relation $\le$ on $\mathcal{P}$ by $(Y_1, \gamma_1) \le (Y_2, \gamma_2)$ if and only if $Y_1 \subseteq Y_2$ and $\gamma_2|_{Y_1} = \gamma_1$. Then it is easy to check that $\mathcal{P}$ with the relation $\le$ is a poset.

We are now going to use Zorn's Lemma to prove $\mathcal{P}$ has a maximal element with respect to this relation. To this end, let $\mathcal{C} = \{(Y_i, \gamma_i) | \, i \in I\}$ be a chain in $\mathcal{P}$. Let $Y = \cup_{i \in I} Y_i$ and $\gamma = \cup_{i \in I} \gamma_i$, where we consider $\gamma_i \subseteq Y_i \times \mathbb{R}$. Since $\mathcal{C}$ is a totally ordered set, Y is a subspace of X and $\gamma : Y \to \mathbb{R}$ is a linear map. Further, for $x \in Y$, there exists $i \in I$ such that $x \in Y_i$. Since $(Y_i, \gamma_i) \in \mathcal{P}$, we have that $\gamma_i(x) \le p(x)$. Then, since $\gamma_{Y_i} = \gamma_i$, we have that $\gamma(x) = \gamma_i(x) \le p(x)$ and so $(Y, \gamma) \in \mathcal{P}$. Clearly, (Y, γ) is an upper bound for $\mathcal{C}$. Thus, by Zorn's Lemma, there exists a maximal element (Z, α) of $\mathcal{P}$.

The proof will be complete if we show that $Z = X$. Suppose not. Let $x_0 \in X \setminus Z$. Define $Z_1 = \{z + t x_0 | \, z \in Z \text{ and } t \in \mathbb{R}\}$. Clearly Z_1 is a subspace of X and $Z \subset Z_1$. Let $z_1, z_2 \in Z$. Then,

$$\begin{aligned} \alpha(z_1) + \alpha(z_2) &= \alpha(z_1 + z_2) \\ &\le p(z_1 + z_2) \\ &\le p(z_1 - x_0) + p(z_2 + x_0). \end{aligned}$$

So,

$$\begin{aligned} \alpha(z_1) - p(z_1 - x_0) &\le p(z_2 + x_0) - \alpha(z_2) \\ &\le \inf_{z_2 \in Z} \left(p(z_2 + x_0) - \alpha(z_2) \right). \end{aligned}$$

Thus,

$$\sup_{z_1 \in Z} (\alpha(z_1) - p(z_1 - x_0)) \leq \inf_{z_2 \in Z} (p(z_2 + x_0) - \alpha(z_2)).$$

Pick $s \in \mathbb{R}$ such that

$$\sup_{z_1 \in Z} (\alpha(z_1) - p(z_1 - x_0)) \leq s \leq \inf_{z_2 \in Z} (p(z_2 + x_0) - \alpha(z_2)).$$

Define $\alpha_1 : Z_1 \to \mathbb{R}$ by $\alpha_1(z + tx_0) = \alpha(z) + ts$. Clearly, $\alpha_1|_Z = \alpha$. Further, since $s \leq \inf_{z_2 \in Z} (p(z_2 + x_0) - \alpha(z_2))$, for any $z \in Z$, we have that $\alpha(z) + s \leq p(z + x_0)$. So, for $t > 0$,

$$\begin{aligned} \alpha_1(z + tx_0) &= \alpha(z) + ts \\ &= t\left(\alpha\left(\frac{z}{t}\right)\right) + s\Big) \\ &\leq tp\left(\frac{z}{t} + x_0\right) \\ &= p(z + tx_0). \end{aligned}$$

Similarly, since $\sup_{z_1 \in Z} (\alpha(z_1) - p(z_1 - x_0)) \leq s$, for any $z \in Z$, we have that $\alpha(z) - s \leq p(z - x_0)$. So, for any $t < 0$,

$$\begin{aligned} \alpha_1(z + tx_0) &= \alpha(z) + ts \\ &= -t\left(\alpha\left(-\frac{z}{t}\right) - s\right) \\ &\leq -tp\left(-\frac{z}{t} - s\right) \\ &= p(z + ts). \end{aligned}$$

Thus, $\alpha(z + tx_0) \leq p(z + tx_0)$ for all $z \in Z$ and $t \in \mathbb{R}$. Hence, $(Z, \alpha) \leq (Z_1, \alpha_1)$ and $Z \neq Z_1$ contradicting the maximality of (Z, α). Hence, we must have that $Z = X$ and the proof is complete. □

We are now ready to prove a version of The Hahn-Banach Extension Theorem.

Theorem 8.3.2. (Hahn-Banach Extension Theorem) *Let $(X, \|\cdot\|)$ be a normed space over the field $\mathbb{K}$ and let X_0 be a subspace of X. Suppose $\phi_0 \in X_0^*$. Then there exists $\phi \in X^*$ such that $\phi|_{X_0} = \phi_0$ and $\|\phi\| = \|\phi_0\|$.*

Proof. **Case 1:** First, suppose $\mathbb{K} = \mathbb{R}$.

Define $p : X \to \mathbb{R}$ by $p(x) = \|\phi_0\|\|x\|$. Clearly p satisfies properties (i) and (ii) of Lemma 8.3.1 and $\phi_0(x) \leq |\phi_0(x)| \leq \|\phi_0\|\|x\| = p(x)$ for all $x \in X_0$. Thus, by Lemma 8.3.1, there exists $\phi : X \to \mathbb{R}$ such that $\phi|_{X_0} = \phi_0$ and $\phi(x) \leq p(x)$ for all $x \in X$.

Let $x \in X$. Then, for $k = 1, 2$,

$$|\phi(x)| = (-1)^k \phi(x) = \phi((-1)^k x) \leq p(x) = \|\phi_0\|\|(-1)^k x\| = \|\phi_0\|\|x\|.$$

Thus, $\phi \in X^*$ and $\|\phi\| \leq \|\phi_0\|$. Further,

$$\begin{aligned}\{|\phi_0(x)| : x \in X_0\,,\ \|x\| \leq 1\} &= \{|\phi(x)| : x \in X_0\,,\ \|x\| \leq 1\} \\ &\subseteq \{|\phi(x)| : x \in X\,,\ \|x\| \leq 1\}\end{aligned}$$

and so $\|\phi_0\| \leq \|\phi\|$. Thus, $\|\phi_0\| = \|\phi\|$.

Case 2: Suppose $\mathbb{K} = \mathbb{C}$.

Let $\alpha_0(x) = \Re(\phi_0(x))$, for all $x \in X_0$ and let $\beta_0(x) = \Im(\phi_0(x))$ for all $x \in X_0$. Note that for all $x \in X_0$, we have

$$\phi_0(ix) + i\beta_0(ix) = \phi_0(ix) = i\phi_0(x) = -\beta_0(x) + i\phi_0(x).$$

Hence, $\beta_0(x) = -\alpha_0(ix)$, for all $x \in X_0$.

Now, if we $\alpha_0 : X_0 \to \mathbb{R}$ is a linear map, where we consider X_0 to be a real vector space by restricting scalar multiplication to $\mathbb{R}$. Further,

$$|\alpha_0(x)| \leq |\phi_0(x)| \leq \|\phi_0\|\|x\|, \text{ for all } x \in X_0.$$

Thus, by Case 1, there exists $\alpha : X \to \mathbb{R}$, where we restrict scalar multiplication on X to $\mathbb{R}$, such that $\alpha|_{X_)} = \alpha$ and $|\alpha(x)| \leq \|\phi_0\|\|x\|$, for all $x \in X$.

Next, define $\phi : X \to \mathbb{C}$ by $\phi(x) = \alpha(x) - i\alpha(ix)$ and note that since $\beta_0(x) = -\alpha_0(ix)$, for all $x \in X_)$, we have that $\phi|_{X_)} = \phi_0$. Now, let $x \in X$. Then $\phi(x) = re^{i\theta}$, for some $r \geq 0$ and $\theta \in \mathbb{R}$. Then,

$$|\phi(x)| = e^{-i\theta}\phi(x) = \phi(e^{-i\theta}x) = \alpha(e^{-i\theta}x) \leq |\alpha(e^{-i\theta}x)| \leq \|\phi_0\|\|e^{-i\theta}x\| = \|\phi_0\|\|x\|.$$

Thus, $\phi \in X^*$ and $\|\phi\| \leq \|\phi_0\|$. Lastly,

$$\begin{aligned}\{|\phi_0(x)| : x \in X_0\,,\ \|x\| \leq 1\} &= \{|\phi(x)| : x \in X_0\,,\ \|x\| \leq 1\} \\ &\subseteq \{|\phi(x)| : x \in X\,,\ \|x\| \leq 1\}\end{aligned}$$

so we also have that $\|\phi_0\| \leq \|\phi\|$. □

Exercise 8.3.3. Let $(X, \|\cdot\|)$ be a normed space over the field $\mathbb{K}$.

(i) Prove X^* separates points of X. *Hint: Let* $x, y \in X$ *where* $x \neq y$ *and let* $X_0 = Span\{x, y\}$. *Define* $f_0 : X_0 \to \mathbb{K}$ *by* $f_0(\alpha x + \beta y) = \alpha$. *Then* $f_0 \in X_0^*$. *Now, use The Hahn-Banach Extension Theorem to find* $f \in X^*$ *such that* $f(x) \neq f(y)$.

(ii) Let $x \in X$. Prove there exists $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$. *Hint: Let* $X_0 = Span\{x\}$ *and define* $f_0 \in X_0^*$ *by* $f_0(\alpha x) = \alpha\|x\|$. *Use The Hahn Banach Extension Theorem to find* $f \in X^*$ *such that* $\|f\| \leq 1$ *and* $f(x) = \|x\|$. *Then, the fact that* $\|f\| \leq 1$ *and* $f(x) = \|x\|$ *implies that actually* $\|f\| = 1$.

8.4 Completions of Normed Spaces

Recall from Section 7.1, to find the completion of a normed space $(X, \|\cdot\|_1)$, it is enough to find an isometric embedding f of $(X, \|\cdot\|_1)$ into a Banach space $(Y, \|\cdot\|_2)$ such that $f(X)$ is dense in Y. It should be stated that "isometric embedding" is typically defined differently in the field of functional analysis. We will discuss this more later.

Definition 8.4.1. Let $(X, \|\cdot\|)$ be a normed space and define a function $j : X \to X^{**}$ by

$$j(x)(f) = f(x), \quad \text{for all } f \in X^* \text{ and } x \in X.$$

The map j is referred to as the **canonical embedding of X into its bidual**.

The above definition takes a few moments to make sense. The map j maps elements of X to elements of X^{**}. Thus, for $x \in X$, we have that $j(x) \in X^{**}$ and X^{**} is the set of all bounded linear functionals from X^* to $\mathbb{K}$. Thus, $j(x) : X^* \to \mathbb{K}$. So, $j(x)$ must map elements of X^* to elements of $\mathbb{K}$. Elements of X^* are bounded linear functionals of the form $f : X \to \mathbb{K}$. Hence, $j(x)$ must map objects like f to elements of $\mathbb{K}$. The most obvious way to do this is to define $j(x)(f) = f(x)$, which is precisely what the map j does.

Theorem 8.4.2. *Let $(X, \|\cdot\|)$ be a normed vector space and let $j : X \to X^{**}$ be the canonical embedding of X into its bidual. Then j is an isometric embedding.*

Proof. Although j is called an "embedding" we have not actually shown that it is injective. To this end, let $x_1, x_2 \in X$ and suppose $j(x_1) = j(x_2)$. Then $j(x_1)(f) = j(x_2)(f)$ for all $f \in X^*$. That is, $f(x_1) = f(x_2)$ for all $f \in X^*$. By Exercise 8.3.3, we know that X^* separates points of X so this can only be the case if $x_1 = x_2$. Hence, j is injective.

To show j is an isometry, we want to show that $\|j(x)\| = \|x\|$ for all $x \in X$. Let $x \in X$. For $f \in X^*$ such that $\|f\| \leq 1$, we have that

$$|j(x)(f)| = |f(x)| \leq \|f\|\|x\| \leq \|x\|.$$

Hence, $\|x\|$ is an upper bound for $\{|j(x)(f)| : f \in X^* \text{ and } \|f\| \leq 1\}$ and so $\|j(x)\| \leq \|x\|$. By Exercise 8.3.3, there exists $f \in X^*$ such that $\|f\| \leq 1$ and $f(x) = \|x\|$. Then, $|j(x)(f)| = |f(x)| = \|x\|$. Thus, $\|x\| \in \{|j(x)(f)| : f \in X^* \text{ and } \|f\| \leq 1\}$ and so $\|x\| \leq \|j(x)\|$. Hence, $\|j(x)\| = \|x\|$ and so j is an isometry. Therefore, j is an isometric embedding of X into X^{**}. □

Since j is an isometric embedding, X is isometric to $j(X)$. Thus, the completion of X, which we denote by $\widehat{X}$, is given by $\widehat{X} = \overline{j(X)}$, since $\overline{j(X)}$ is a closed subset of a complete metric space. It should be mentioned though, that the normed space $(X, \|\cdot\|)$ is not just a topological space but an algebraic space as well. We can't simply identify the normed space X with $j(X)$ because they are isometric (which relies soley on the metric space structure) without also checking that they are algebraically equivalent as well. Fortunately, j is also a linear operator, that is, a vector space homomorphism (checking this fact is a following exercise). So it preserves the algebraic structure as well. Thus, for all intents and purposes, X and $j(X)$ are the same normed vector space. For this reason, "isometric embedding" is typically defined differently in a functional analysis textbook as we also want the isometric embedding to be a linear operator to preserve the operations.

Exercise 8.4.3. Let $(X, \|\cdot\|)$ be a normed space and let $j : X \to X^{**}$ be the canonical embedding of X into its bidual. Prove that j is a linear operator.

8.5 The Weak and Weak* Topologies

We are now able to discuss other types of topologies on normed vector spaces.

Definition 8.5.1. Let $(X, \|\cdot\|)$ be a normed vector space. The **weak topology** on X is the weak topology generated by the set of functions X^*. That is, the weak topology is the weakest topology on X which makes all the elements of X^* continuous. Further, the **weak* topology** on X^* is the weak topology generated by the set of functions $j(X)$. That is, it is the weakest topology on X^* which makes all the elements of $j(X)$ continuous.

Note that we can only define the weak topology on X but we can define the weak topology and the weak* topology on X^*. In general, $j(X) \neq X^{**}$, thus the weak topology on X^* is not the same thing as the weak* topology on X^*. Obviously, since $j(X) \subseteq X^{**}$, we have that the weak* topology is weaker than the weak topology on X^*. When we have a normed space X such that $j(X) = X^{**}$, we call it **reflexive**.

Notice that the weak* topology on X^* is precisely the topology of pointwise convergence. That is, a net $(f_\lambda)_{\lambda\in\Lambda}$ converges to some $f \in X^*$ if and only if $j(x)(f_\lambda) \to j(x)(f)$, for all $x \in X$, which is if and only if $f_\lambda(x) \to f(x)$ for all $x \in X$. Thus, a motivation for defining the weak* topology on X^* is made clear.

In general, normed vector spaces endowed with the weak topology or weak* topology are not first countable. Thus, we have to use nets rather than sequences to check properties like openness, closedness, and compactness of sets or continuity of functions. If you recall, an important property for the generating set of functions for a weak topology is that it separates points. We check that this is the case for the weak and weak* topologies in the next exercise.

Exercise 8.5.2. Let $(X, \|\cdot\|)$ be a normed vector space. We have already seen that X^* separates points of X. Prove that $j(X)$ separates points of X^*.

Thus, the weak topology and the weak* topology are always Hausdorff.

It turns out that when investigating topological vector spaces, the situation is very different if we are dealing with a finite-dimensional vector space versus an infinite-dimensional vector space. So different, in fact, that the two situations fall under the umbrella of different fields of mathematics, with the study of finite-dimensional vector spaces belonging to Linear Algebra, whereas, the study of infinite-dimensional vector spaces is the focus of Functional Analysis. In the next section, we will focus our attention on finite-dimensional vector spaces. Afterwards, we will focus on infinite-dimensional vector spaces. Once these sections are complete, the reader should see why these two topics are markedly different.

8.6 Finite-Dimensional Vector Spaces

We should already be aware of the fact, from Linear Algebra, that if X is a finite-dimensional real vector space, with dimension n, then X is algebraically equivalent to $\mathbb{R}^n$. Further, if X is an n-dimensional complex vector space, then X is algebraically equivalent to $\mathbb{C}^n$. Indeed, given an n-dimensional vector space X with basis $B = \{e_1, e_2, \ldots, e_n\}$, every element x of X can be uniquely expressed in the form $x = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n$, for some $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{K}$. Thus, we can make the association

$$x = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n \longleftrightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \alpha_n \end{bmatrix}$$

which respects the algebraic operations. Thus, if we are only interested in the algebraic properties of finite-dimensional vector spaces, then there is no need to study anything other than $\mathbb{R}^n$ for real vector spaces and $\mathbb{C}^n$ for complex vector spaces.

What if, though, we want to incorporate topological properties as well? Are there different vector topologies that we can impose on $\mathbb{R}^n$ and $\mathbb{C}^n$ which will give us different topological properties? In this section, we will answer these questions.

First, recall that when we say two metrics defined on the same set are equivalent, we mean that the topologies the two metrics generate are the same. Here, when we say two norms defined on the same set are equivalent, we mean the topologies the two norms generate are equivalent. We also saw that if d_1 and d_2 are two metrics defined on a set X and there exists $k, K >)$ such that

$$kd_1(x_1, x_2) \leq d_2(x_1, x_2) \leq Kd_1(x_1, x_2), \quad \text{for all } x_1, x_2 \in X$$

then d_1 and d_2 are equivalent. If our metrics are given by the norms $\| \cdot \|_1$ and $\| \cdot \|_2$ then the above string of inequalities is equivalent to

$$k\|x\|_1 \leq \|x\|_2 \leq K\|x\|_1, \quad \text{for all } x \in X.$$

Thus, two norms $\|c\|_1$ and $\| \cdot \|_2$ defined on the same set X are equivalent (or generate the same topology) if we can find $k, K > 0$ such that

$$k\|x\|_1 \leq \|x\|_2 \leq K\|x\|_1, \quad \text{for all } x \in X.$$

We are now ready to look at the first theorem of this section.

Theorem 8.6.1. *Let $\| \cdot \|_1$ and $\| \cdot \|_2$ be norms defined on the vector space $\mathbb{K}^n$ over the field $\mathbb{K}$, for some $n \in \mathbb{Z}_+$. Then there exists $k, K > 0$ such that*

$$k\|x\|_1 \leq \|x\|_2 \leq K\|x\|_1, \quad \textit{for all } x \in \mathbb{K}^n.$$

Thus the norms $\| \cdot \|_1$ and $\| \cdot \|_2$ are equivalent.

Proof. Let $\beta = \{b_1, b_2, \ldots, b_n\}$ be a basis for $\mathbb{K}^n$. Define $\| \cdot \|_3 : \mathbb{K}^n \to \mathbb{R}$ by $\|x\|_3 = \sum_{k=1}^{n} |\alpha_k|$, where $x = \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n$. Since representations of vectors as linear combinations of basis elements are unique, we have that $\| \cdot \|_3$ is well-defined. It is easy to check that $\| \cdot \|_3$ is a norm on $\mathbb{K}^n$.

Now, consider the norm $\|\cdot\|_1$ as a function $\|\cdot\|_1 : (\mathbb{K}^n, \|\cdot\|_3) \to [0,\infty)$. Let $M = \max\{\|b_k\|_1 : k = 1, 2, \dots, n\}$. Then, for $x \in \mathbb{K}^n$, where $x = \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n$, we have

$$\|x\|_1 = \left\|\sum_{k=1}^{n} \alpha_k b_k\right\| \leq M \sum_{k=1}^{n} |\alpha_k| = M\|x\|_3.$$

Thus, $\|\cdot\|_1 : (\mathbb{K}^n, \|\cdot\|_3) \to [0,\infty)$ is a continuous function.

Next, we would like to show that $S = \{x \in \mathbb{K}^n | \, \|x\|_3 = 1\}$ is a compact subset of $(\mathbb{K}^n, \|\cdot\|_3)$. Since $(\mathbb{K}^n, \|\cdot\|_3)$ is a metric space, by Theorem **??**, it is enough to show S is sequentially compact in $(\mathbb{K}^n, \|\cdot\|_3)$. Let $(\overline{x}_k)_{k=1}^{\infty}$ be a sequence in $(\mathbb{K}^n, \|\cdot\|_3)$. We have to show there exists a subsequence which coinverges to an element of S. Let $\overline{x}_k = \alpha_{k1} b_1 + \alpha_{k2} b_2 + \cdots + \alpha_{kn} b_n$, for all $k \in \mathbb{Z}_+$. Since

$$|\alpha_{k1} \leq \sum_{j=1}^{n} |\alpha_{kj}| = \|\overline{x}_k\|_3 = 1$$

we have that $(\alpha_{k1})_{k=1}^{\infty}$ is a sequence in $[-1,1]$. Since the set $[-1,1]$ is compact, there exists a subsequence $(\alpha_{k_j 1})_{j=1}^{\infty}$ such that $\alpha_{k_j 1} \to \gamma_1$, for some $\gamma_1 \in [-1,1]$. Similarly, we can find a subsequence of $(\alpha_{k_j 2})_{j=1}^{\infty}$ which converges to some $\gamma_2 \in [-1,1]$, and continue in this fashion up to n so that we then have a subsequence $(\overline{x}_{k_s})_{s=1}^{\infty}$ of $(\overline{x}_k)_{k=1}^{\infty}$, where $\overline{x}_{k_s} = \alpha_{k_s 1} b_1 + \alpha_{k_s 2} b_2 + \cdots + \alpha_{k_s n} b_n$, such that $\overline{x}_{k_s} \xrightarrow{\|\cdot\|_3} \overline{x}$, where $\overline{x} = \gamma_1 b_1 + \gamma_2 b_2 + \cdots + \gamma_n b_n$. And, since S is closed with respect to $\|\cdot\|_3$, we have that $\overline{x} \in S$. Thus, S is compact with respect to $\|\cdot\|_3$.

Now, we have that $\|\cdot\|_1 : (\mathbb{K}^n, \|\cdot\|_3) \to [0,\infty)$ is continuous and S is a compact subset of $(\mathbb{K}^n, \|\cdot\|_3)$. Thus, $\|\cdot\|_1(S)$ is a compact subset of $[0,\infty)$. Hence, by The Extreme Value Theorem (Theorem **??**), $\|\cdot\|_1$ attains a minimum c_1 and maximum c_2 on S. Note that $c_1 \neq 0$ since $\overline{0} \notin S$. Thus, we have $c_1, c_2 > 0$ such that $c_1 \leq \|x\|_1 \leq c_2$, for all $x \in S$.

Next, let $x \in \mathbb{K}^n$. Then $\frac{x}{\|x\|_3} \in S$ so

$$c_1 \leq \left\| \frac{x}{\|x\|_3} \right\|_1 \leq c_2$$

and so

$$c_1\|x\|_3 \leq \|x\|_1 \leq c_2\|x\|_3.$$

Since $\|\cdot\|_1$ is a generic norm on X, as is $\|\cdot\|_2$, we also have shown that there exists $d_1, d_2 > 0$ such that

$$d_1\|x\|_3 \leq \|x\|_2\|d_2\|x\|_3.$$

Then,

$$\|x\|_1 \leq c_2\|x\|_3 \leq \frac{c_2}{d_1}\|x\|_2$$

and

$$\|x\|_1 \geq c_1\|x\|_3 \geq \frac{c_1}{d_2}\|x\|_2.$$

Hence, there exists $k, K > 0$ such that

$$k\|x\|_2 \leq \|x\|_1 \leq K\|x\|_2$$

and the proof is complete. □

Thus, any norm we define on $\mathbb{R}^n$ will produce the same topology as the Euclidean norm which, in turn, produces the usual topology. Similarly, any norm we define on $\mathbb{C}^n$ will produce the usual topology on $\mathbb{C}^n$. Thus, for all intents and purposes, the only n-dimensional real normed vector space is the Banach space $\mathbb{R}^n$ with the Euclidean norm and the only n-dimensional complex normed vector space is the Banach space $\mathbb{C}^n$ with the Euclidean norm.

For general normed spaces $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$, we know that $\mathcal{B}(X,Y) \subseteq L(X,Y)$. The next proposition tells us, that if X is finite-dimensional, then we actually have

$\mathcal{B}(X, Y) = L(X, Y)$.

Proposition 8.6.2. *Let* $(X, \|\cdot\|_1)$ *and* $(Y, \|\cdot\|_2)$ *be normed spaces over the same field* $\mathbb{K}$. *If* X *is finite-dimensional, then* $\mathcal{B}(X, Y) = L(X, Y)$.

Proof. Since X is finite-dimensional, there exists $n \in Z_+$ and $b_1, b_2, \cdots, b_n \in X$ such that $\beta = \{b_1, b_2, \ldots, b_n\}$ is a basis for X. Recall from the proof of Theorem 8.6.1, if we define $\|\cdot\|_3 : X \to \mathbb{R}$ by $\|x\|_3 = \sum_{k=1}^{n} |\alpha_k|$, where $x = \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n$, then $\|\cdot\|_3$ is a norm on X. Thus, from Theorem 8.6.1, there exists $C > 0$ such that $\|x\|_3 \leq C\|x\|_1$ for all $x \in X$.

Let $T \in L(X, Y)$. Now, let $M = \max\{\|Tb_k\|_2 : \, k = 1, 2, \ldots, n\}$ and let $x \in X$ such that $\|x\| \leq 1$. Then, $x = \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_2 b_n$, for some $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{K}$. Further,

$$\begin{aligned} \|Tx\|_2 &= \|T(\alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n)\|_2 \\ &= \|\alpha_1 T b_1 + \alpha_2 T b_2 + \cdots + \alpha_n T b_n\|_2 \\ &\leq |\alpha_1| \|Tb_1\|_2 + |\alpha_2| \|Tb_2\|_2 + \cdots + |\alpha_n| \|Tb_n\|_2 \\ &\leq M\left(|\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|\right) \\ &= M\|x\|_3 \\ &\leq MC\|x\|_1 \\ &\leq MC. \end{aligned}$$

Thus, the set $\{\|Tx\|_2 : \, x \in X \, , \, \|x\| \leq 1\}$ is bounded above and so $T \in \mathcal{B}(X, Y)$. □

Recall that $L(\mathbb{R}^n, \mathbb{R}^m) = M_{m \times n}(\mathbb{R})$. Since we now know $L(\mathbb{R}^n, \mathbb{R}^m) = \mathcal{B}(\mathbb{R}^n, \mathbb{R}^m)$, we have that all matrices are automatically bounded, i.e., continuous.

Further, if $X = \mathbb{R}^n$, then $X^* = \mathcal{B}(\mathbb{R}^n, \mathbb{R}) = M_{1 \times n}(\mathbb{R})$. From Linear Algebra, we know that $M_{1 \times n}(\mathbb{R})$ is an n-dimensional vector space, thus X^* with the operator norm is algebraically and topologically equivalent to $\mathbb{R}^n$. That is, $X^* = X$. Consequently, we also have that X is reflexive since this implies $X^{**} = X$.

Still, we defined the weak topology for general normed spaces X (the weak topology generated by X^*). What is the weak topology on $\mathbb{R}^n$ or $\mathbb{C}^n$? The next theorem answers this question.

Theorem 8.6.3. *Let τ_1 we the weak topology on $\mathbb{K}^n$ and let τ_2 be the topology on $\mathbb{K}^n$ generated by the Euclidean norm. Then $\tau_1 = \tau_2$.*

Proof. Let $X = \mathbb{K}^n$. By definition, every element of X^* is continuous with respect to τ_2. Since τ_1 is the weakest topology which makes every element of X^* continuous, we have that $\tau_1 \subseteq \tau_2$.

For the other inclusion, consider the projection maps $\pi_k : X \to \mathbb{K}$, for $k = 1, 2, \ldots, n$. It is easy to check that each projection map is linear. By Proposition 8.6.2, we have that $\pi_k \in X^*$, for each $k = 1, 2, \ldots, n$. Thus, by Exercise **??**, if τ_3 is the weak topology generated by the projections maps, then $\tau_3 \subseteq \tau_1$. The weak topology generated by the projection maps is precisely the product topology on X. Further, for $X = \mathbb{K}^n$, we know the product topology equals the usual topology τ_2. Thus, $\tau_2 = \tau_3 \subseteq \tau_1$ and therefore $\tau_1 = \tau_2$. □

Thus, there is no need to consider the weak topology on a finite-dimensional vector space as it still produces the usual topology. If X is finite-dimensional, we saw that $X = X^*$, thus, we could impose the weak* topology on X but, if $Y = X^*$, then $Y^* = X^{**} = X^*$ so the above theorem tells us that the weak* topology on $\mathbb{R}^n$ is again going to be the usual topology.

In conclusion, for finite-dimensional vector spaces, the only meaningful vector topology to consider is the usual topology. Further, all of our linear operators are matrices which are automatically continuous, making any discussion about continuity of linear functions unnecessary.

While we almost exclusively use the Euclidean norm as the norm of choice for $\mathbb{K}^n$ (because it has additional geometric properties which are useful), this is not always the case with the finite-dimensional vector space $M_{m\times n}(\mathbb{K})$. While all of the norms defined on $M_{m\times n}(\mathbb{K})$ generate the same topology, it is sometimes useful to use norms different than the operator norm.

8.7 Infinite-Dimensional Vector Spaces

We saw in the last section that if the domain of a linear operator was finite-dimensional, then the linear operator was automatically bounded. This is not the case, if the domain is an infinite-dimensional vector space as the next example illustrates.

Example 8.7.1. Let $P([0,1])$ be the vector space of polynomials defined on $[0,1]]$. Since $P([0,1])$ is a subspace of $C([0,1])$, we can define the supremum norm $\|\cdot\|_\infty$ on $P([0,1])$. Define $D : P([0,1]) \to P([0,1])$ by $D(p) = p'$, where p' is the derivative of p. Clearly, D is a linear operator but it is not bounded. To see this, consider $p_n(x) = x^n$, for all $n \in \mathbb{Z}_+$. Then, $\|p_n\|_\infty = 1$ but $\|D(p_n)\|_\infty = n$. Thus, for any $M > 0$, we can find a $p_n \in P([0,1])$ such that $\|D(p_n)\|_\infty \geq M\|p_n\|_\infty$. Hence, D is not bounded (and so not continuous).

The above example shows that, in general, linear operators are not automatically bounded. Next, we will look at some important theorems about linear operators on general normed vector spaces.

It is easy to verify (exercise below) that if $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ are normed spaces and $T \in \mathcal{B}(X,Y)$ is an open function, then T is surjective.

Exercise 8.7.2. Let $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ be normed spaces and let $T \in \mathcal{B}(X,Y)$. If T is an open map, then T is surjective. *Hint: We want to prove $T(X) = Y$. Obviously, $T(X) \subseteq Y$. For the other inclusion, ince T is open, $T(X)$ is open.*

Hence, there exists $\epsilon > 0$ *such that* $B_Y(0, \epsilon) \subseteq T(X)$*. Now show, for any* $y \in Y$*, that* $y \in T(X)$*.*

The next theorem proves the converse of the above exercise in the case when $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ are Banach spaces.

Theorem 8.7.3. (Open Mapping Theorem) *Let* $(X, \|\cdot\|_1)$ *and* $(Y, \|\cdot\|_2)$ *be Banach spaces and let* $T \in \mathcal{B}(X, Y)$*. If* T *is surjective, then* T *is an open function.*

Proof. Let $B_Z(x_0, r) = \{x \in Z | \, \|x - x_0\| < r\}$, where Z is either X or Y.

Claim: There exists $\epsilon > 0$ such that $B_Y(0, \epsilon) \subseteq T(B_X(0, 1))$.

Since T is surjective, we have that $Y = \cup_{n=1}^{\infty} T(B_X(0, n))$. Since Y is a complete metric space, by The Baire Category Theorem (Theorem **??**) there must exist $n \in \mathbb{Z}_+$ such that $T(B_X(0, n))$ is not nowhere dense. This is, $\left(\overline{T(B_X(0, n))}\right)^o \neq \emptyset$. Since $\overline{T(B_X(0, n))} = n\overline{T(B_x(0, 1))}$, we then have that $\left(\overline{T(B_X(0, 1))}\right)^o \neq \emptyset$. Hence, there exists $y \in Y$ and $s > 0$ such that $B_Y(y, s) \subseteq \overline{T(B_X(0, 1))}$. Note that we also have $B_Y(-y, s) \subseteq \overline{T(B_X(0, 1))}$. Thus, if $y_0 \in Y$ and $\|y_0\| < s$, then,

$$y_0 = y + (y_0 - y) \in 2\overline{T(B_X(0, 1))} \subseteq \overline{T(B_X(0, 2))}$$

and so $B_Y(0, s) \subseteq \overline{T(B_X(0, 2))}$. Hence, we have that, for $t = \frac{s}{2}$, the inclusion $B_Y(0, t) \subseteq \overline{T(B_X(0, 1))}$ and so, for $\epsilon > 0$, we have $B_Y(0, \epsilon t) \subseteq \overline{T(B_X(0, \epsilon))}$.

Thus, we have shown the following property:

Property: For all $\epsilon > 0$ and all $\delta > 0$, if $y \in Y$ and $\|y\|_2 < \epsilon t$, then there exists $x \in X$ such that $\|x\|_1 < \epsilon$ and $\|y - Tx\|_2 < \delta$.

Let $y_0 \in B_Y(0, t)$. Then, using the above property for $\epsilon = 1$, $\delta = \frac{t}{2}$, and $y = y_0$, we find $x_0 \in X$ such that $\|x_0\|_1 < 1$ and $\|y_0 - Tx_0\|_2 < \frac{t}{2}$.

Now, use the property again but with $\epsilon = \frac{1}{2}$, $\delta = \frac{t}{2^2}$, and $y = y_0 - Tx_0$ (where we then label $y_1 = y_0 - Tx_0$) to find $x_1 \in X$ such that $\|x_1\|_1 < \frac{1}{2}$ and $\|y_1 - Tx_1\|_2 < \frac{t}{2^2}$.

Continue in this fashion to construct a sequence $(x_n)_{n=0}^{\infty}$ in X and a sequence $(y_n)_{n=1}^{\infty}$ in Y such that $\|x_n\|_1 < \frac{1}{2^n}$, $\|y_n\|_2 < \frac{t}{2^n}$, and $y_n = y_{n-1} - Tx_{n-1}$, for all $n \in \mathbb{Z}_+$.

It is an exercise given afterward to show that since $\sum_{n=0}^{\infty} \|x_n\|_1 < \infty$ and X is a complete metric space, that $\sum_{n=0}^{\infty} = x$, for some $x \in X$. Also, note that for $n \in Z_+$,

$$\begin{aligned} y_n &= y_{n-1} - Tx_{n-1} \\ &= y_{n-2} - Tx_{n-2} - Tx_{n-1} \\ &= \dots \\ &= y_0 - T\left(\sum_{k=0}^{n-1} x_k\right). \end{aligned}$$

Thus, since T is continuous,

$$0 = \lim_{n\to\infty} y_n = \lim_{n\to\infty}\left(y_0 - T\left(\sum_{k=0}^{n} x_k\right)\right) = y_0 - Tx$$

and so $Tx = y_0$. Further, since $\|\sum_{k=0}^{n} x_k\|_1 < 2$, for all $n \in \mathbb{N}$ and $\sum_{k=0}^{n} x_k \to x$, we have that $\|x\|_1 < 3$. Thus, $B_Y(0,t) \subseteq T(B_X(0,3))$ and so, for $\epsilon = \frac{t}{3}$, we have $B_Y(0,\epsilon) \subseteq T(B_x(0,1))$. This completes the proof of the claim.

We now want to show $T(B_X(0,1))$ is open. Let $y \in T(B_X(0,1))$. Then $y = Tx$, for some $x \in X$ where $\|x\|_1 < 1$. Pick $r \in \mathbb{R}$ such that $0 < r < 1 - \|x\|_1$. Then,

$$B_Y(y, r\epsilon) = y + B_Y(0, r\epsilon) \subseteq y + T(B_X(0,r)) = T(B_X(x,r)) \subseteq T(B_X(0,1))$$

and so $T(B_X(0,1))$ is open.

Now, for any basic open set $B_X(x_0,r)$ in X, we have $T(B_X(x_0,r)) = Tx_0 + rT(B_X(0,1))$

which is open. Hence, T is an open function. □

Exercise 8.7.4. Let $(X, \|\cdot\|)$ be a Banach space and let $(x_n)_{n=1}^{\infty}$ be a sequence in X. Prove that if $\sum_{n=1}^{\infty} \|x_n\| < \infty$, then there exists $x \in X$ such that $\sum_{n=1}^{\infty} x_n = x$. *Hint: Recall your definitions from Calculus. When we say $\sum_{n=1}^{\infty} x_n = x$, we mean the sequence of partial sums $(\sum_{n=1}^{k} x_n)_{k=1}^{\infty}$ converges to x. Since $\sum_{n=1}^{\infty} \|x_n\| < \infty$, we know its sequence of partial sums is Cauchy. Use this to show the sequence of partial sums $(\sum_{n=1}^{k} x_n)_{k=1}^{\infty}$ is Cauchy. Then use the fact that X is a complete metric space.*

One consequence of The Open Mapping Theorem is given in the following corollary.

Corollary 8.7.5. *Let $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ be Banach spaces and let $T \in \mathcal{B}(X, Y)$. If T is bijective, then T is invertible and $T^{-1} \in \mathcal{B}(Y, X)$.*

Proof. Obviously, T^{-1} is well-defined, since T is bijective. Since T is surjective, we have from The Open Mapping Theorem, that T is open. Thus, T^{-1} is continuous. Since linear operators are continuous if and only if they are bounded, we have that $T^{-1} \in \mathcal{B}(Y, X)$. □

As with most functions, linear operators included, we are often interested in knowing when they are invertible. We already know that if a function is bijective then it has an inverse. Thus, given a bijective $T \in \mathcal{B}(X, Y)$, we already know that T^{-1} exists. If X and Y are finite-dimensional, then we automatically obtain that T^{-1} is bounded, since all linear operators in this case are bounded. What the above proposition tells us is that, in the case when X and Y are infinite-dimensional, there is still no need to check if $T^{-1} \in \mathcal{B}(Y, X)$ as The Open Mapping Theorem guarantees that this is the case.

Another important theorem about the continuity of a linear operator has to do with the **graph** of a linear operator. We will start with a definition.

Definition 8.7.6. Let X and Y be vector spaces and let $T \in L(X,Y)$. The **graph of T, denoted by $G(T)$, is a subset of $X \times Y$ and is given by**

$$G(T) = \{(x, Tx) \,|\, x \in X\}.$$

Exercise 8.7.7. Let $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ be normed spaces. Prove that if $T \in \mathcal{B}(X,Y)$, then $G(T)$ is closed.

The next theorem proves the converse of the above exercise in the case when both normed spaces are complete.

Theorem 8.7.8. (Closed Graph Theorem) *Let $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ be Banach spaces and let $T \in L(X,Y)$. If T has a closed graph (that is, $G(T)$ is a closed subset of $X \times Y$ with the product topology), then $T \in \mathcal{B}(X,Y)$.*

Proof. Define $\|\cdot\|_3 : X \times Y \to \mathbb{R}$ by $\|(x,y)\|_3 = \|x\|_1 + \|y\|_2$. It is easy to check that $\|\cdot\|_3$ is a norm on $X \times Y$. Further, a sequence $((x_n, y_n))_{n=1}^{\infty}$ in $X \times Y$ converges to some $(x,y) \in X \times Y$ with respect to $\|\cdot\|_3$ if and only if $x_n \to x$ and $y_n \to y$. This, in turn, is true, if and only if $\pi_1((x_n,y_n)) \to \pi_1((x,y))$ and $\pi_2((x_n,y_n)) \to \pi_2((x,y))$ which is if and only if $(x_n, y_n) \to (x,y)$ with respect to the product topology. Thus, the norm $\|\cdot\|_3$ induces the product topology on $X \times Y$. Since $G(T)$ is closed, we have that $(G(T), \|\cdot\|_3)$ is a Banach space. The maps π_1 and π_2 are continuous since $X \times Y$ has the product topology, thus $\pi_1|_{G(T)}$ is continuous by Theorem **??**. Further, $\pi_1|_{G(T)}$ is bijective and so, by Corollary 8.7.5, we have that $(\pi_1|_{G(T)})^{-1}$ is continuous. Thus, $T = \pi_2 \circ (\pi_1|_{G(T)})^{-1}$ is continuous by Exercise **??**. □

At first glance, the usefulness of The Closed Graph Theorem might be missed. Suppose $T \in L(X,Y)$ and we would like to prove that T is continuous. The standard approach is to let $(x_n)_{n=1}^{\infty}$ be a sequence in X such that $x_n \to x$, for some $x \in X$, and then we are tasked with showing that $Tx_n \to Tx$.

The Closed Graph Theorem tells us that to prove T is continuous, it is enough to prove $G(T)$ is closed. To do this, we let $((x_n, Tx_n))_{n=1}^{\infty}$ be a sequence in $G(T)$ such that $(x_n, Tx_n) \to (x, y)$, for some $(x, y) \in X \times Y$, and we want to show that $(x, y) \in G(T)$. To do this, we simply have to show $Tx = y$. Note though that since $(x_n, Tx_n) \to (x, y)$, we have that $x_n \to x$ and $Tx_n \to y$.

Hence, in our first strategy we end up with a sequence $(x_n)_{n=1}^{\infty}$ such that $x_n \to x$, for some $x \in X$, and we have to show that $(Tx_n)_{n=1}^{\infty}$ converges to something, and that the something is Tx.

In our second strategy, provided to us by The Closed Graph Theorem, we end up with a sequence $(x_n)_{n=1}^{\infty}$ such that $x_n \to x$, for some $x \in X$ *and* we further have that $Tx_n \to y$, for some $y \in Y$, and we have to show $Tx = y$. Thus, with the second strategy, one of our tasks in the first strategy is done for us. We already know $(Tx_n)_{n=1}^{\infty}$ converges to something. We just have to show that the something is Tx.

If the reader recalls the proof of Theorem 8.6.1, which showed that any two norms defined on a finite-dimensional vector space are equivalent, it relied heavily on the fact that the closed unit sphere, S, was compact. If the closed unit sphere of an infinite-dimensional vector space is also compact, a variation of the proof given in Theorem 8.6.1 might show that two norms defined on the same infinite-dimensional vector space X must be equivalent, especially if we allow ourselves to add a few assumptions about X. One of the surprising things about infinite-dimensional Banach spaces is that, in fact, the closed unit sphere is *never* compact, as the next theorem shows.

Theorem 8.7.9. *If $(X, \|\cdot\|)$ is a Banach space. The unit sphere $S = \{x \in X \mid \|x\| = 1\}$ is compact with respect to the norm if and only if X is finite-dimensional.*

Proof. For the backwards direction, suppose X is finite-dimensional. Since S is closed and bounded, by The Heine-Borel Theorem, S is compact.

For the forward direction, suppose S is compact. For any $x \in X$ we have, by Exercise 8.3.3, that there exists $f_x \in X^*$ such that $f_x(x) = \|x\|$. Thus, for $x \neq 0$, we have that $x \in X \setminus f_x^{-1}(\{0\})$. Since $\{0\}$ is closed and f_x is continuous, the set $f_x^{-1}(\{0\})$ is closed and so $X \setminus f_x^{-1}(\{0\})$ is open. Thus,

$$S \subseteq \bigcup_{f \in X^*} \left(X \setminus f^{-1}(\{0\})\right)$$

is an open cover of S. Since S is compact, there exists a finite subcover. That is, there exists $n \in Z_+$ and $f_1, f_2, \ldots, f_n \in X^*$ such that

$$S \subseteq \bigcup_{k=1}^{n} \left(X \setminus f_k^{-1}(\{0\})\right). \tag{8.1}$$

Define $T : X \to \mathbb{K}^n$ by $T(x) = (f_1(x), f_2(x), \ldots, f_n(x))$. The map T is clearly linear. Further, if $x_1, x_2 \in X$ and $T(x_1) = T(x_2)$ then $f_k(x_1) = f_k(x_2)$ for all $k = 1, 2, \ldots, n$. If $x_1 \neq x_2$, then $\frac{x_1 - x_2}{\|x_1 - x_2\|} \in S$ and

$$f\left(\frac{x_1 - x_2}{\|x_1 - x_2\|}\right) = 0, \text{ for all } k = 1, 2, \ldots, n.$$

But, by Equation (8.1), we have that there exists $k = 1, 2, \ldots, n$ such that $\frac{x_1 - x_2}{\|x_1 - x_2\|} \in X \setminus f_k^{-1}(\{0\})$, which implies that $f_k(\frac{x_1 - x_2}{\|x_1 - x_2\|}) \neq 0$. This is a contradiction. Hence, we have that $x_1 = x_2$ and so T is injective. Therefore, since $T : X \to \mathbb{K}^n$ is linear and injective, the dimension of X must be at most n and so X is finite-dimensional. □

Exercise 8.7.10. Let $(X, \|\cdot\|)$ be an infinite dimensional Banach space.

(i) We know X^* is a Banach space. Prove that it is also infinite-dimensional.

(ii) Prove that $S = \{f \in X^* : \|f\| = 1\}$ is closed.

(iii) Prove that the closed unit ball $\overline{B}_{X^*}(0, 1)$ in X^* is not compact. *Hint: If the closed unit ball were compact, what would that imply about the closed set S?*

Note that the above theorem does not imply that given an infinite-dimensional vector space, two different norms can't induce the same topology. It is simply saying, among other things, that our proof strategy to show norms must be equivalent in finite dimensions won't work here. The following example shows that we can, in fact, have two norms defined on the same vector space which do not produce the same topology.

Example 8.7.11. Consider the vector space $C([0,1])$. Define $\|f\|_\infty = \sup\{|f(x)| : x \in [0,1]\}$ which is well-defined since a continuous function $f : [0,1] \to \mathbb{R}$ is bounded. Also, define $\|f\|_1 = \int_0^1 |f(x)|dx$ which is well-defined since continuous functions are Riemann integrable. Consider $f_n : [0,1] \to \mathbb{R}$ defined by $f_n(x) = x^n$. Then $f_n \xrightarrow{\|\cdot\|_1} 0$, where 0 is the constant 0 function, since

$$\|f_n - 0\|_1 = \int_0^1 x^n dx = \frac{1}{n+1} \to 0.$$

But, $(f_n)_{n=1}^\infty$ does not converge to 0 with respect to $\|\cdot\|_\infty$ since $\|f_n - 0\|_\infty = 1$ for all $n \in \mathbb{Z}_+$. Thus, by Proposition **??**, the topology generated by $\|\cdot\|_\infty$ is not contained in the topology generated by $\|\cdot\|_1$ and so the two norms generate different topologies.

While in finite dimensions, any norm we define will produce the usual topology, thus ensuring the exact same topological properties. For infinite-dimensional vector spaces, different norms will potentially produce different topologies, thus giving us different topological properties. So, for example, a subset A of an infinite-dimensional vector space X could be closed or not closed, compact or not compact, etc. depending on the norm that we define on X.

For finite-dimensional vector spaces, we saw that the weak and weak* topologies were also equal to the usual topology on $\mathbb{R}^n$ or $\mathbb{C}^n$. In general, for infinite-dimensional normed spaces, the weak topology and (when applicable) the weak* topology will produce different vector topologies than the one induced by the norm.

In summary, given a finite-dimensional vector space, it does not matter what norm we define on the space or whether we instead impose the weak or weak* topology on the space, we will always end up with the usual topology on $\mathbb{R}^n$ or $\mathbb{C}^n$. We will also have that all of our linear operators are matrices which are automatically continuous.

For infinite-dimensional vector spaces, different norms will potentially produce different topologies giving us a seemingly endless list of possible normed vector spaces to study. Further, we can impose the weak topology and (when applicable) weak* topology on these same normed spaces to provide us with an entirely different topological space to consider. To illustrate the richness of topological vector spaces and usefulness of considering different topologies on the same vector space, consider the dual X^* of an infinite-dimensional normed space X. While we saw in Exercise 8.7.10 that the closed unit ball in X^*, equipped with the operator norm, is not compact. The closed unit ball in X^* is compact, if we equip X^* with the weak* topology.

Theorem 8.7.12. (Alaoglu's Theorem) *Let $(X, \|\cdot\|)$ be a normed vector space over the field $\mathbb{K}$. Then the closed unit ball of X^* is compact with respect to the weak* topology.*

Proof. Consider the set $Z = \prod_{x\in X} \overline{B}_{\mathbb{K}}(0, \|x\|)$ with the product topology. By Tychonoff's Theorem (Theorem **??**), the set Z is compact. Note that for any $f \in Z$, we have that $|f(x)| \leq \|x\|$, for all $x \in X$. The elements of Z are not necessarily linear though. Thus, the elements of $\overline{B}_{X^*}(0,1)$ are precisely the elements of Z which are linear. Also, we saw in Chapter 4 that the product topology is precisely the topology of pointwise convergence and we know the weak* topology on X^* is also the topology of pointwise convergence. Thus, the weak* topology on $\overline{B}_{X^*}(0,1)$, as a subspace of Z, is precisely the topology of pointwise convergence. Thus, it suffices to show that $\overline{B}_{X^*}(0,1)$ is closed with respect to pointwise convergence. Then, by Exercise **??**, we have that $\overline{B}_{X^*}(0,1)$ is compact. Let $(f_\lambda)_{\lambda\in\Lambda}$ be a net in $\overline{B}_{X^*}(0,1)$ such that $f_\lambda \xrightarrow{w^*} f$, where $f \in X^*$. We know that $f \in Z$, since Z is compact. Thus, it suffices to show that f is linear. Let $x_1, x_2 \in X$ and let $\alpha \in \mathbb{K}$. Since $f_\lambda \xrightarrow{w^*} f$, we have that $f_\lambda(x_1) \to f(x_1)$, $f_\lambda(x_2) \to f(x_2)$, and $f_\lambda(x_1 + \alpha x_2) \to f(x_1 + \alpha x_2)$. Thus,

$$\begin{aligned} f(x_1 + \alpha x_2) &= \lim_{\lambda \to \infty} f_\lambda(x_1 + \alpha x_2) \\ &= \lim_{\lambda \to \infty} (f_\lambda(x_1) + \alpha f_\lambda(x_2)) \\ &= \lim_{\lambda \to \infty} f_\lambda(x_2) + \alpha \lim_{\lambda \to \infty} f_\lambda(x_2) \\ &= f(x_1) + \alpha f(x_2). \end{aligned}$$

Thus, f is linear and so $\overline{B}_{X^*}(0,1)$ is a compact set with respect to the weak* topology.

□

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