MATH 422-Introduction to Topology

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January 27, 2026

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Metric Spaces Revisited

Let us first recall everything that we have learned about metric spaces since Chapter 2. Let (X, d) be a metric space.

In Chapter 3, we saw that $\beta = \{B_d(x, r) | \, x \in X, \, r > 0\}$ is a base for a topology on X which we refer to as the topology on X induced by d. Although, admittedly, we always assumed this is the topology that we are using on a metric space so we often omitted this language. We will continue to do so in the remaining chapters.

In Chapter 4, we saw that if $A \subseteq X$, then the subspace topology on A is precisely the topology on A induced by the metric d_0, where $d_0 = d|_{A \times A}$.

In Chapter 5, we proved that all metric spaces are first countable so we can use sequences to show sets are open or closed, or to show that functions are continuous.

In Chapter 6, we proved that all metric spaces are Hausdorff spaces (which implies, among other things, that convergent sequences have unique limits). We also saw that compact metric spaces must be closed and bounded and that sequential compactness is equivalent to compactness.

In this chapter, we want to investigate more properties of metric spaces and look at some important theorems about metric spaces.

7.1 Complete Metric Spaces

Definition 7.1.1. Let (X, d) be a metric space and let $(x_n)_{n=1}^{\infty}$ be a sequence in X. We say the sequence $(x_n)_{n=1}^{\infty}$ is a **Cauchy sequence** if, for all $\epsilon > 0$, there exists $N \in \mathbb{Z}_+$ such that, if $n, m \geq N$, then $d(x_n, x_m) < \epsilon$.

Intuition tells us that Cauchy sequences are very much related to convergent sequences. Are they the same thing? Surely they are not, or we would not have wasted our time with an unnecessary definition. The next theorem tells us that

Cauchy sequences are more general than convergent sequences.

Theorem 7.1.2. *Every convergent sequence in a metric space is a Cauchy sequence.*

Proof. Let (X, d) be a metric space and let $(x_n)_{n=1}^{\infty}$ be a convergent sequence. Thus, there exists $x \in X$ such that $x_n \xrightarrow{d} x$. Let $\epsilon > 0$. Since $x_n \xrightarrow{d} x$, there exists $N \in \mathbb{Z}_+$ such that, for all $n \geq N$, we have $d(x_n, x) < \frac{\epsilon}{2}$. Thus, for $n, m \geq N$, we have

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

which completes the proof. □

As mentioned above, while convergent sequences are Cauchy sequences, not all Cauchy sequences are convergent as the next few examples illustrate.

Example 7.1.3. (i) Consider the interval $X = (0, \infty)$ with the usual metric on $\mathbb{R}$. Define the sequence $(x_n)_{n=1}^{\infty}$ by $x_n = \frac{1}{n}$ for all $n \in \mathbb{Z}_+$. Then $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence. Indeed, let $\epsilon > 0$. Pick $N > \frac{1}{\epsilon}$. Then, for $n, m \geq N$, assume, without loss of generality, that $n \geq m$. Then,

$$d(x_n, x_m) = \left| \frac{1}{n} - \frac{1}{m} \right| = \frac{n-m}{nm} \leq \frac{n}{nm} = \frac{1}{m} \leq \frac{1}{N} < \epsilon.$$

However, $(x_n)_{n=1}^{\infty}$ is not convergent as there exists no $x \in X$ such that $x_n \xrightarrow{d} x$.

(ii) Consider $X = \mathbb{Q}$ and define the sequence $(x_n)_{n=1}^{\infty}$ in X by

$$x_1 = 1 \quad , \quad x_2 = 1.4 \quad , \quad x_3 = 1.41 \quad , \quad x_4 = 1.414 \quad , \textit{etc.}$$

where we remind the reader that $\sqrt{2} = 1.41421356\ldots$. Then it is easy to check that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence. Indeed, given $\epsilon > 0$, pick $N \in \mathbb{Z}_+$ such that $\frac{1}{10^{N-1}} < \epsilon$. Let $n, m \geq N$. Without loss of generality, suppose $n \geq m$. Then, for some digit d,

$$d(x_n, x_m) \leq d(x_m, x_{m+1}) \leq \frac{d}{10^m} \leq \frac{d}{10^N} \leq \frac{1}{10^{N-1}} < \epsilon.$$

Again, though, there exists no $x \in \mathbb{Q}$ such that $x_n \xrightarrow{d} x$.

(iii) Consider $X = \mathbb{R}$ and define a metric d on $\mathbb{R}$ by $d(x, y) = |e^{-x} - e^{-y}|$. Then the sequence $(x_n)_{n=1}^\infty$ defined by $x_n = n$ for all $n \in \mathbb{Z}_+$ is a Cauchy sequence which does not converge. The details are left as a following exercise.

Exercise 7.1.4. Let $X = \mathbb{R}$ and define a metric d on $\mathbb{R}$ by $d(x, y) = |e^{-x} - e^{-y}|$. Define the sequence $(x_n)_{n=1}^\infty$ in X by $x_n = n$ for all $n \in \mathbb{Z}_+$. First, prove that $(x_n)_{n=1}^\infty$ is Cauchy. Then, prove that there exists no $x \in X$ such that $x_n \xrightarrow{d} x$.

As the reader perhaps noticed, examples (i) and (ii) above feel a bit cheap. Of course, we could always take a convergent sequence in $\mathbb{R}$, define the usual metric d on a set A which contains the sequence but does not contain the limit of the sequence. We would then produce a metric space (A, d) which has a Cauchy sequence which does not converge (simply by omitting its limit from A). In this case, the metric space (A, d) has Cauchy sequences which don't converge simply because we omitted their limits. That is, the set A just wasn't made big enough or isn't *complete*. This is the intuition behind the next definition. Of course, the situation is actually more complicated than this as Example (iii) illustrates. There we have a Cauchy sequence where no obvious limit is being omitted from $\mathbb{R}$. In examples (i) and (ii) we are also getting ahead of ourselves slightly as the need for Cauchy sequences in $\mathbb{Q}$ to converge is what lead to one of the original definitions of $\mathbb{R}$ (more on this later).

Definition 7.1.5. Let (X, d) be a metric space. If every Cauchy sequence in (X, d) converges, then we say (X, d) is **complete** or refer to (X, d) as a complete metric space.

For those who have taken an abstract algebra course, completeness of metric spaces for analysts is somewhat analogous to operations being closed for algebraic structures. An algebraic structure can be difficult to study when we add or multiply two elements and the result is no longer within the structure. A similar situation is true in analysis. Since sequences are such important tools for analysts, having analytic structures where the limits of sequences, in some sense, fall outside of our structure, is not preferred.

We will start our investigation of complete metric spaces by first proving $\mathbb{R}$ with the usual metric is complete. First, we need a lemma.

Lemma 7.1.6. *Let (X, d) be a metric space and let $(x_n)_{n=1}^{\infty}$ be a sequence in X. If $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence, then $(x_n)_{n=1}^{\infty}$ is bounded.*

Proof. Let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence in X. Then there exists $N \in \mathbb{Z}_+$ such that, for all $n, m \geq N$, we have that $d(x_n, x_m) < 1$. In particular, for all $m \geq N$, we have that $d(x_N, x_m) < 1$. Now, let $r = \max\{d(x_N, x_k) \,|\, k = 1, \ldots, N-1\} + 1$. Then, $\{x_n \,|\, n \in \mathbb{Z}_+\} \subseteq B_d(x_N, r)$ and so $(x_n)_{n=1}^{\infty}$ is bounded. □

Theorem 7.1.7. (Cauchy Criterion) *Let d be the usual metric on $\mathbb{R}^n$. Then $(\mathbb{R}^n, d)$ is a complete metric space.*

Proof. We must prove that given a Cauchy sequence $(x_n)_{n=1}^{\infty}$, there exists some $x \in \mathbb{R}^n$ such that $x_n \to x$. To this end, let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence in $\mathbb{R}^n$. From Lemma 7.1.6, we have that $(x_n)_{n=1}^{\infty}$ is bounded. Thus, by The Bolzano-Weierstrass Theorem (see Theorem **??**), the sequence $(x_n)_{n=1}^{\infty}$ has a convergent subsequence $(x_{n_k})_{k=1}^{\infty}$. Hence, there exists $x \in \mathbb{R}^n$ such that $x_{n_k} \to x$. Our goal is to show that actually $(x_n)_{n=1}^{\infty}$ converges to x.

Let $\epsilon > 0$. Since $(x_n)_{n=1}^{\infty}$ is Cauchy, there exists $N \in \mathbb{Z}_+$ such that, for all $n, m \geq N$, we have that $d(x_n, x_m) < \frac{\epsilon}{2}$. Since $x_{n_k} \to x$, there exists $K \in \mathbb{Z}_+$ such that, for all $k \geq K$, we have $d(x_{n_k}, x) < \frac{\epsilon}{2}$.

Since $(n_k)_{k=1}^\infty$ is a strictly increasing sequence, there exists $k_0 \in \mathbb{Z}_+$ such that $n_{k_0} \geq N$. Pick k_0 large enough so that $k_0 \geq K$. Then, for $n \geq N$,

$$\begin{aligned} d(x_n, x) &\leq d(x_n, x_{n_{k_0}}) + d(x_{n_{k_0}}, x) \\ &< \frac{\epsilon}{2} + d(x_{n_{k_0}}, x) && \text{since } n, n_{k_0} \geq N \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} && \text{since } k_0 \geq K \\ &= \epsilon. \end{aligned}$$

Thus, $x_n \to x$ and so $(x_n)_{n=1}^\infty$ is a convergent sequence. □

If we investigate the proof above, there is nothing we used about the metric space $(\mathbb{R}^n, d)$ specifically besides The Bolzano-Weierstrass Theorem. Thus, if a general metric space has the property that every bounded sequence has a convergent subsequence, then it must be complete. This gives us the following corollary.

Corollary 7.1.8. *Let (X, d) be a metric space. Suppose every bounded sequence in X has a convergent subsequence. Then (X, d) is a complete metric space.*

Proof. The proof is a following exercise. □

Exercise 7.1.9. Prove Corollary 7.1.8. *Hint: Follow the proof of Theorem 7.1.7.*

The next proposition is fairly obvious but it is used quite often so it is worth giving formally. Recall that for a metric space (X, d) and a subset A of X, the subspace topology on A is the same as the topology generated by $d|_{A\times A}$.

Proposition 7.1.10. *Let (X, d) be a complete metric space and let $A \subseteq X$. If A is closed, then $(A, d|_{A\times A})$ is a complete metric space.*

Proof. Let $(x_n)_{n=1}^\infty$ be a Cauchy sequence in A. We want to show $(x_n)_{n=1}^\infty$ converges to some $x \in A$. Since $(x_n)_{n=1}^\infty$ is Cauchy in A, then it is also Cauchy in X. Since

(X, d) is a complete metric space, there exists $x \in X$ such that $x_n \xrightarrow{d} x$. Since $(x_n)_{n=1}^{\infty}$ is a sequence in A, we have that x is a limit point of A. Since A is closed, we then have that $x \in A$. Therefore, $(A, d|_{A\times A})$ is complete. □

As we discussed after Examples 7.1.3, often times, a metric space is not complete simply because we omitted some of the limit points from the space. If our metric space is a subspace of a complete metric space, then Proposition 7.1.10 gives us an easy make to turn our metric space into a complete metric space by taking the closure. To be precise, if (X, d) is a complete metric space and $A \subseteq X$, while $(A, d|_{A\times A})$ might not be complete, $(\overline{A}, d|_{\overline{A}\times\overline{A}})$ is complete and $(A, d|_{A\times A})$ is a subspace of $(\overline{A}, d|_{\overline{A}\times\overline{A}})$.

For instance, in Example 7.1.3, part (i), we saw that $X = (0, \infty)$, with the usual metric, is not complete. It is, however, a subspace of the complete metric space $\mathbb{R}$ with the usual metric. Thus, $\overline{X} = [0, \infty)$ is a complete metric space which contains X. Similarly, in part (ii) of Example 7.1.3, we saw that $\mathbb{Q}$ with the usual metric was not complete. It is, however, a subspace of the complete metric space $\mathbb{R}$ with the usual metric. Proposition 7.1.10 then tells us that $\overline{\mathbb{Q}}$ is a complete metric space containing $\mathbb{Q}$. In this case though, we have to include all of $\mathbb{R}$ since $\overline{\mathbb{Q}} = \mathbb{R}$.

If we examine part (iii) of Example 7.1.3 though, where we defined the metric d on $\mathbb{R}$ by $d(x, y) = |e^{-x} - e^{-y}|$, there is no obvious complete metric space which has $(\mathbb{R}, d)$ as a subspace. Can we still find a complete metric space $(X, \widehat{d})$ which contains $(\mathbb{R}, d)$ as a subspace (and so we would need $d = \widehat{d}|_{\mathbb{R}\times\mathbb{R}}$)? Further, would $X = \overline{\mathbb{R}}^{\widehat{d}}$ as it did in parts (i) and (ii)? Our next goal is to answer these questions. We first have to discuss isometries between metric spaces.

Definition 7.1.11. Let (X, d_1) and (Y, d_2) be metric spaces and let $f : (X, d_1) \to (Y, d_2)$. We say f is an **isometry** if

$$d_2(f(x_1), f(x_2)) = d_1(x_1, x_2) \quad , \text{ for all } x_1, x_2 \in X.$$

An informal way to define an isometry between metric spaces is to say that it is a function which preserves the distance between two points. Having a bijective isometry between two metric spaces is a much stronger property than having a homeomorphism between the metric spaces when we view the metric spaces as topological spaces whose topologies are generated by their respective metrics. This is the topic of the next proposition.

Proposition 7.1.12. *Let (X, d_1) and (Y, d_2) be metric spaces and let $f : (X, d_1) \to (X, d_2)$. If f is a bijective isometry, then $(X, d_1) \cong (Y, d_2)$.*

Proof. Let $f : (X, d_1) \to (Y, d_2)$ be a bijective isometry. We want to show f is also a homeomorphism. We already have that f is a bijection, so it suffices to prove f and f^{-1} are continuous.

To show f is continuous, we will use Theorem **??**. Let $(x_n)_{n=1}^{\infty}$ be a sequence in X and suppose $x_n \to x$, for some $x \in X$. Let $\epsilon > 0$. Since $x_n \to x$, there exists $N \in \mathbb{Z}_+$ such that, for all $n \geq N$, we have that $d_1(x_n, x) < \epsilon$. Then, for $n \geq N$,

$$d_2(f(x_n), f(x)) = d_1(x_n, x) < \epsilon.$$

Thus, $f(x_n) \to f(x)$.

Similarly, to show f^{-1} is continuous, let $(y_n)_{n=1}^{\infty}$ be a sequence in Y such that $y_n \to y$, for some $y \in Y$. Since f is onto, there exists $x_k \in X$ such that $f(x_k) = y_k$ for all $k \in \mathbb{Z}_+$ and there exists $x \in X$ such that $f(x) = y$. We wish to show that $f^{-1}(y_n) \to f^{-1}(y)$. That is, we wish to show $f^{-1}(f(x_n)) \to f^{-1}(f(x))$, i.e., $x_n \to x$. Let $\epsilon > 0$. Since $y_n \to y$, there exists $N \in \mathbb{Z}_+$ such that, for all $n \geq N$, we have $d_2(y_n, y) < \epsilon$. Then, for $n \geq N$,

$$d_1(x_n, x) = d_2(f(x_n), f(x)) = d(y_n, y) < \epsilon$$

and so $x_n \to x$.

Therefore, f is a homeomorphism between (X, d_1) and (Y, d_2). □

Notice something we showed in the proof above was that if f is an isometry then f is continuous. We will give this as a corollary.

Corollary 7.1.13. *Let (X, d_1) and (Y, d_2) be metric spaces. If $f : (X, d_1) \to (Y, d_2)$ is an isometry, then f is continuous.*

Definition 7.1.14. Let (X, d_1) and (Y, d_2) be metric spaces. If there exists a bijective isometry from (X, d_1) to (Y, d_2), we say (X, d_1) and (Y, d_2) are **isometric** and we write $(X, d_1) \equiv (Y, d_2)$.

We this notation, we can now restate Proposition 7.1.12 as follows: If (X, d_1) and (Y, d_2) are metric spaces and $(X, d_1) \equiv (Y, d_2)$, then $(X, d_1) \cong (Y, d_2)$.

It's rather obvious but it is worth pointing out, we use the symbol $\equiv$ to denote two spaces being isometric because $\equiv$ defines an equivalence relation on the collection of all metric spaces.

Metric spaces being isometric is the metric space analogue to isomorphic groups or rings in abstract algebra or homeomorphic topological spaces in general topology. That is to say, for all intents and purposes, isometric metric spaces are essentially the same mathematical object if we consider only their metric space properties. The proposition above tells us that isometric metric spaces will also be essentially the same mathematical object if we consider only their topological properties as well.Thus, any topological property which is preserved by homeomorphisms is also preserved by bijective isometries. The next exercise illustrates that properties specific to metric spaces are preserved by bijective isometries.

Exercise 7.1.15. Let (X, d_1) and (Y, d_2) be isometric metric spaces. If (X, d_1) is complete, then (Y, d_2) is complete.

Notice that an isometry (not necessarily bijective) is automatically injective (this is a following exercise), thus, if $f : (X, d_1) \to (Y, d_2)$ is an isometry, then $f : (X, d_1) \to (f(X), d_2|_{f(X)\times f(X)})$ is a bijective isometry, and so, $(X, d_1) \equiv (f(X), d_2|_{f(X)\times f(X)})$, where $(f(X), d_2|_{f(X)\times f(X)})$ is a subspace of (Y, d_2). Hence, (X, d_1) is essentially a subspace of (Y, d_2). We often summarize this fact by simply saying that the metric space (X, d_1) *embeds isometrically* into the metric space (Y, d_2). Let us formalize this language in the next definition once we give the above mentioned exercise.

Exercise 7.1.16. Let (X, d_1) and (Y, d_2) be metric spaces. If $f : (X, d_1) \to (Y, d_2)$ is an isometry, then f is injective.

Definition 7.1.17. Let (X, d_1) and (Y, d_2) be metric spaces. If there exists an isometry $f : (X, d_1) \to (Y, d_2)$, then we say (X, d_1) **embeds isometrically** into the metric space (Y, d_2) and we call the map f an **isometric embedding** of (X, d_1) into (Y, d_2).

Just to reiterate what was said earlier, if (X, d_1) embeds isometrically into (Y, d_2), then this means (X, d_1) is isometric to a subspace of (Y, d_2) (specifically, the subspace $(f(X), d_2|_{f(X)\times f(X)})$).

Now, to return to our question from earlier. That is, if we have a metric space (X, d_1) which is not complete and is not contained in an obvious complete metric space, can we find a way to "complete" the metric space (X, d_1) anyway? The isometric embeddings give us one way to do this. If we have an isometric embedding $f : (X, d_1) \to (Y, d_2)$, where (Y, d_2) is a complete metric space, then we can identify the metric space (X, d_1) with the metric space $(f(X), d_2|_{f(X)\times f(X)})$ and then we can "complete" (X, d_1) by considering it as a subspace of $(\overline{f(X)}, d_2|_{\overline{f(X)}\times\overline{f(X)}})$ which is a complete metric space since it is a closed subspace of (Y, d_2). This leads us to the following definition.

Definition 7.1.18. Let (X, d_1) be a metric space and let (Y, d_2) be a complete metric space. If $f : (X, d_1) \to (Y, d_2)$ is an isometric embedding of (X, d_1) into (Y, d_2), then the subspace $\overline{f(X)}$, with the subspace topology, is called the **completion** of (X, d_1).

From our previous discussions, we can now say that if (X, d) is a metric space which embeds isometrically, via the isometric embedding f, into a complete metric space, then we can complete X by identifying it with $f(X)$ (which it is isometric to) and then take the closure of $f(X)$. So, we now have a way to complete a metric space if we have an isometric embedding of that metric space into a complete metric space. Thus, the remaining question is, "Can any metric space be isometrically embedded into a complete metric space?" The answer is "yes," as the next theorem shows. We first need a lemma.

Lemma 7.1.19. *Let (X, d) be a metric space and suppose A is a dense subset of X. If every Cauchy sequence in A converges to an element of X, then (X, d) is complete.*

Proof. Let (X, d) be a metric space and let $A \subseteq X$ such that $\overline{A} = X$. Further, suppose that every Cauchy sequence in A converges to an element of X. Let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence in X.

Let $n \in \mathbb{Z}_+$. Since $\overline{A} = X$, there exists $a_n \in A$ such that $d(a_n, x_n) < \frac{1}{n}$. Thus, we have defined a sequence $(a_n)_{n=1}^{\infty}$. We want to show $(a_n)_{n=1}^{\infty}$ is Cauchy. To this end, let $\epsilon > 0$. Since $(x_n)_{n=1}^{\infty}$ is Cauchy, there exists $N_0 \in \mathbb{Z}_+$ such that, for all $n, m \geq N_0$, we have that $d(x_n, x_m) < \frac{\epsilon}{3}$. Pick $N \in \mathbb{Z}_+$ such that $N \geq N_0$ and $N \geq \frac{3}{\epsilon}$. Let $m, n \geq N$. Then,

$$\begin{aligned} d(a_n, a_m) &\leq d(a_n, x_n) + d(x_n, x_m) + d(x_m, a_m) \\ &< \frac{1}{n} + \frac{\epsilon}{3} + \frac{1}{m} \\ &\leq \frac{1}{N} + \frac{\epsilon}{3} + \frac{1}{N} \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

Thus, $(a_n)_{n=1}^\infty$ is Cauchy. By our assumption, there exists $x \in X$ such that $a_n \to x$. To finish the proof, we want to show that $x_n \to x$.

Let $\epsilon > 0$. Since $a_n \to x$, there exists $N_0 \in \mathbb{Z}_+$ such that, for all $n \geq N_0$, we have that $d(a_n, x) < \frac{\epsilon}{2}$. Pick $N \in \mathbb{Z}_+$ such that $N \geq N_0$ and $N \geq \frac{2}{\epsilon}$. Then, for $n \geq N$,

$$d(x_n, x) \leq d(x_n, a_n) + d(a_n, x) < \frac{1}{N} + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence, $x_n \to x$ and so (X, d) is complete. □

Theorem 7.1.20. *Let (X, d_1) be a metric space. Then there exists a complete metric space (Y, d_2) and an isometric embedding $f : (X, d_1) \to (Y, d_2)$. Further, $f(X)$ is dense in Y.*

Proof. Let (X, d_1) be a metric space. Define $\widehat{X}$ to be the set of all Cauchy sequences in X. Define a relation $\sim$ on $\widehat{X}$ by $(x_n)_{n=1}^\infty \sim (y_n)_{n=1}^\infty$ if and only if $d_1(x_n, y_n) \to 0$. Proving $\sim$ is an equivalence relation on $\widehat{X}$ is a following exercise.

Let $[(x_n)_{n=1}^\infty]$ denote the equivalence class of $(x_n)_{n=1}^\infty$ and let $Y = {}^{X}\!/\!\sim$ be the set of all such equivalence classes. Define $d_2 : Y \times Y \to \mathbb{R}$ by

$$d_2([(x_n)_{n=1}^\infty], [(y_n)_{n=1}^\infty]) = \lim_{n\to\infty} d_1(x_n, y_n).$$

We first have to prove that d_2 is well-defined. First, we have to show that given any $[(x_n)_{n=1}^\infty], [(y_n)_{n=1}^\infty] \in Y$, we have $d_2([(x_n)_{n=1}^\infty], [(y_n)_{n=1}^\infty]) \in \mathbb{R}$. We will do this by first showing the sequence $(d_1(x_n, y_n))_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$. To this end, let $\epsilon > 0$. Since $(x_n)_{n=1}^\infty$ is Cauchy, there exists $N_1 \in \mathbb{Z}_+$ such that, for all $m, n \geq N_1$, we have that $d_1(x_m, x_n) < \frac{\epsilon}{2}$. Similarly, since $(y_n)_{n=1}^\infty$ is Cauchy, there exists $N_2 \in \mathbb{Z}_+$ such that, for all $m, n \geq N_2$, we have $d_1(y_m, y_n) < \frac{\epsilon}{2}$. Let $N \geq \max\{N_1, N_2\}$. Then, for $n, m \geq N$, we have

$$d_1(x_n, y_n) \leq d_1(x_n, x_m) + d_1(x_m, y_m) + d_1(y_m, y_n)$$

so,

$$d_1(x_n, y_n) - d_1(x_m, y_m) \leq d_1(x_n, x_m) + d_1(y_m, y_n).$$

Similarly,

$$d_1(x_m, y_m) \leq d_1(x_m, x_n) + d_1(x_n, y_n) + d_1(y_n, y_m),$$

so,

$$d_1(x_m, y_m) - d_1(x_n, y_n) \leq d_1(x_n, x_m) + d_1(y_m, y_n)$$

and thus,

$$|d_1(x_n, y_n) - d_1(x_m, y_m)| \leq d_1(x_n, x_m) + d_1(y_m, y_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence, $(d_1(x_n, y_n))_{n=1}^{\infty}$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete, there exists $r \in \mathbb{R}$ such that $d_1(x_n, y_n) \to r$. Thus,

$$d_2([(x_n)_{n=1}^{\infty}], [(y_n)_{n=1}^{\infty}]) = \lim_{n\to\infty} d_1(x_n, y_n) = r \in \mathbb{R}.$$

Next, to finish showing d_2 is well-defined, given $(x'_n)_{n=1}^{\infty} \in [(x_n)_{n=1}^{\infty}]$ and $(y'_n)_{n=1}^{\infty} \in [(y_n)_{n=1}^{\infty}]$, we need

$$\lim_{n\to\infty} d_1(x'_n, y'_n) = \lim_{n\to\infty} d_1(x_n, y_n).$$

Since $(x'_n)_{n=1}^{\infty} \in [(x_n)_{n=1}^{\infty}]$, we have that $d(x'_n, x_n) \to 0$ and, since $(y'_n)_{n=1}^{\infty} \in [(y_n)_{n=1}^{\infty}]$, we have $d_1(y'_n, y_n) \to 0$. So,

$$\lim_{n\to\infty} d_1(x'_n, y'_n) \leq \lim_{n\to\infty} (d_1(x'_n, x_n) + d_1(x_n, y_n) + d_1(y_n, y'_n)) = \lim_{n\to\infty} d_1(x_n, y_n).$$

A symmetric argument shows

$$\lim_{n\to\infty} d_1(x_n, y_n) \leq \lim_{n\to\infty} d_1(x'_n, y'_n)$$

and so,

$$\lim_{n\to\infty} d_1(x_n, y_n) = \lim_{n\to\infty} d_1(x'_n, y'_n).$$

Therefore, d_2 is well-defined. To verify d_2 is a metric on Y is a following exercise.

Now, define $h : X \to Y$ by $h(x) = [(x)_{n=1}^{\infty}]$ where $(x)_{n=1}^{\infty} = (x, x, x, \ldots)$. We want to show h is an isometric embedding. Let $x_1, x_2 \in X$. Then

$$d_2(h(x_1), h(x_2)) = d_2([(x_1)_{n=1}^{\infty}, [(x_2)_{n=1}^{\infty}]) = \lim_{n\to\infty} d_1(x_1, x_2) = d_1(x_1, x_2).$$

Hence, h is an isometric embedding. Thus, (X, d_1) embeds isometrically into (Y, d_2).

We now want to show (Y, d_2) is complete. We will first show that $h(X)$ is dense in Y, then show that every Cauchy sequence in $h(X)$ converges to an element of Y. Thus, by Lemma 7.1.19, we will have that (Y, d_2) is complete. First, to show $h(X)$ is dense in Y, let $[(z_n)_{n=1}^{\infty}] \in Y$. Consider the sequence $(h(z_k))_{k=1}^{\infty} = ([(z_k)_{n=1}^{\infty}])_{k=1}^{\infty}$ in Y. Let $\epsilon > 0$. Since $(z_n)_{n=1}^{\infty}$ is Cauchy, there exists $N \in \mathbb{Z}_+$ such that, for all $m, n \geq N$, we have that $d_1(z_n, z_m) < \frac{\epsilon}{2}$. Then, pick $K = N$ and let $k \geq N$. Then,

$$\begin{aligned} d_2\left([(z_k)_{n=1}^{\infty}], [(z_n)_{n=1}^{\infty}]\right) &= \lim_{n\to\infty} d_1(z_k, z_n) \\ &\leq \lim_{n\to\infty} \frac{\epsilon}{2} && \text{for } n \geq K = N \\ &= \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus, $h(z_n) \to [(z_n)_{n=1}^{\infty}]$ and so $h(X)$ is dense in Y. Now, let $(h(x_k))_{k=1}^{\infty}$ be a Cauchy sequence in $h(X)$. Once we show $[(x_n)_{n=1}^{\infty}] \in Y$ then we saw above that $h(x_k) \to [(x_n)_{n=1}^{\infty}]$.

Let $\epsilon > 0$. Since $(h(x_k))_{k=1}^{\infty}$ is Cauchy, there exists $K \in \mathbb{Z}_+$ such that, for all $k, m \geq K$, we have

$$d_1(x_k, x_m) = \lim_{n\to\infty} d_1(x_k, x_m) = d_2([(x_k)_{n=1}^{\infty}], [(x_m)_{n=1}^{\infty}]) = d_2(h(x_k), h(x_m)) < \epsilon.$$

Now, let $N = K$ and suppose $n, m \geq N$. Then,

$$d_1(x_n, x_m) = d_2(h(x_n), h(x_m)) < \epsilon.$$

Thus, $[(x_n)_{n=1}^{\infty}] \in Y$. Therefore, (Y, d_2) is complete. We have already shown that $h(X)$ is dense in Y, so the proof is complete. □

Exercise 7.1.21. (i) Prove that the relation $\sim$ defined in the proof of Theorem 7.1.20 is an equivalence relation.

(ii) Prove that d_2 defined in the proof of Theorem 7.1.20 is a metric on Y.

Thus, every metric space has *a* completion. Notice in Definition 7.1.18, that we say *the* completion instead. This is due to the fact that any two completions of the same metric space must be isometric to each other as the next propositon shows. We first need to prove a useful lemma.

Lemma 7.1.22. *Let (X, d) be a metric space. Suppose $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are sequences in X such that $x_n \to x$ and $y_n \to y$, for some $x, y \in X$. Then, $d(x_n, y_n) \to d(x, y)$.*

Proof. Since $x_n \to x$, there exists $N_1 \in \mathbb{Z}_+$ such that, for all $n \geq N_1$, we have that $d(x_n, x) < \frac{\epsilon}{2}$. Similarly, since $y_n \to y$, there exists $N_2 \in \mathbb{Z}_+$ such that, for all $n \geq N_2$, we have that $d(y_n, y) < \frac{\epsilon}{2}$. Let $N = \max\{N_1, N_2\}$ and suppose $n \geq N$. Then,

$$d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n),$$

so,

$$d(x_n, y_n) - d(x, y) \leq d(x_n, x) + d(y, y_n).$$

Similarly,

$$d(x, y) \leq d(x, x_n) + d(x_n, y_n) + d(y_n, y),$$

so,

$$d(x,y) - d(x_n,y_n) \leq d(x,x_n) + d(y_n,y).$$

Thus,

$$|d(x,y) - d(x_n,y_n)| \leq d(x,x_n) + d(y_n,y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, $d(x_n,y_n) \to d(x,y)$. □

Proposition 7.1.23. *Let (X,d_1) be a metric space and let (Y,d_2) and (Z,d_3) be complete metric spaces. Suppose $f:(X,d_1)\to(Y,d_2)$ and $g:(X,d_1)\to(Z,d_3)$ are both isometric embeddings. Then $(\overline{f(X)}, d_2|_{\overline{f(X)}\times\overline{f(X)}})$ is isometric to $(\overline{g(X)}, d_2|_{\overline{g(X)}\times\overline{g(X)}})$.*

Proof. Let (X,d_1) be a metric space and let (Y,d_2) and (Z,d_3) be complete metric spaces. Further, let $f:(X,d_1)\to(Y,d_2)$ and $g:(X,d_1)\to(Z,d_3)$ are both isometric embeddings. Thus, we have that $f:(X,d_1)\to(f(X),d_2|_{f(X)\times f(X)})$ is a bijective isometry and $g:(X,d_1)\to(g(X),d_3|_{g(X)\times g(X)})$ is a bijective isometry. Thus,

$$h = g\circ f^{-1} : (f(X), d_2|_{f(X)\times f(X)}) \to (g(X), d_3|_{g(X)\times g(X)})$$

is a bijective isometry. Define

$$\widehat{h} : (\overline{f(X)}, d_2|_{\overline{f(X)}\times\overline{f(X)}}) \to (\overline{g(X)}, d_3|_{\overline{g(X)}\times\overline{g(X)}})$$

as follows: for $x\in\overline{f(X)}$, let $(x_n)_{n=1}^{\infty}$ be a sequence in $f(X)$ such that $x_n\to x$ (we can just take $x_n = x$, for all $n\in\mathbb{Z}_+$, if $x\in f(X)$). Since $(x_n)_{n=1}^{\infty}$ converges, it is Cauchy. Since h is an isometry, the sequence $(h(x_n))_{n=1}^{\infty}$ is a Cauchy sequence in $g(X)$. Since $\overline{g(X)}$ is complete, there exists $y_x\in\overline{g(x)}$ such that $h(x_n)\to y_x$. Define $\widehat{h}(x) = y_x$. Note, for $x\in f(X)$, we have that $\widehat{h}(x) = h(x)$.

First, we must show $\widehat{h}$ is well-defined. To check, let $(x_n)_{n=1}^{\infty}$ and $(z_n)_{n=1}^{\infty}$ be sequences in $f(X)$ such that $x_n\to x$ and $z_n\to x$, for some $x\in\overline{f(X)}$. We want to show that $\lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} h(z_n)$. To this end, define the sequence $(w_n)_{n=1}^{\infty}$ in $f(X)$ by $w_n = x_{(n+1)/2}$, if n is odd, and $w_n = z_{n/2}$, if n is even. Since $x_n\to x$ and $z_n\to x$, we have that $w_n \to x$. Hence, $(w_n)_{n=1}^{\infty}$ is a Cauchy sequence in $f(X)$. Since h is an isometry, we have that $(h(w_n))_{n=1}^{\infty}$ is a Cauchy sequence in $g(X)$. Since $\overline{g(X)}$ is complete, there exists $s \in \overline{g(X)}$ such that $h(w_n) \to s$. Since $(h(x_n))_{n=1}^{\infty}$ and $(h(z_n))_{n=1}^{\infty}$ are subsequences of $(h(w_n))_{n=1}^{\infty}$, we have that $h(x_n) \to s$ and $h(z_n) \to s$. Thus, $\lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} h(z_n)$, and so, $\widehat{h}$ is well-defined.

Now, we must show $\widehat{h}$ is a bijective isometry. Let us first show that it is an isometry. Let $x, y \in \overline{f(X)}$ and let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be sequences in $f(X)$ such that $x_n \to x$ and $y_n \to y$. Then, we have that $\widehat{h}(x_n) \to \widehat{h}(x)$ and $\widehat{h}(y_n) \to \widehat{h}(y)$. Then, we have

$$\begin{aligned}
d_3(\widehat{h}(x), \widehat{h}(y)) &= \lim_{n\to\infty} d_3(\widehat{h}(x_n), \widehat{h}(y_n)) && \text{by Lemma 7.1.22} \\
&= \lim_{n\to\infty} d_3(h(x_n), h(y_n)) && \text{since } x_n, y_n \in f(X) \\
&= \lim_{n\to\infty} d_2(x_n, y_n) && \text{since } h \text{ is an isometry} \\
&= d_2(x, y) && \text{by Lemma 7.1.22}
\end{aligned}$$

Thus, $\widehat{h}$ is an isometry.

Since $\widehat{h}$ is an isometry, we know that it is injective, so it is left to prove that it is surjective. Let $y \in \overline{g(X)}$. Then there exists a sequence $(y_n)_{n=1}^{\infty}$ in $g(X)$ such that $y_n \to y$. Since h is surjective, there exists a sequence $(x_n)_{n=1}^{\infty}$ in $f(X)$ such that $h(x_n) = y_n$ for all $n \in \mathbb{Z}_+$. Since $y_n \to y$, we have that $h(x_n) \to y$. Thus, $(h(x_n))_{n=1}^{\infty}$ is a Cauchy sequence in $g(X)$. Since h is an isometry, $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in $f(X)$. Since $\overline{f(X)}$ is complete, there exists $x \in \overline{f(X)}$ such that $x_n \to x$. Thus, $\widehat{h}(x) = \lim_{n\to\infty} h(x_n) = y$, and so $\widehat{h}$ is surjective.

Therefore, we have established that $\widehat{h}$ is a bijective isometry and so $\overline{f(X)}$ and $\overline{g(X)}$ are isometric. □

Thus, to find *the* completion of a metric space (X, d_1), it is enough to find an isometric embedding f of (X, d_1) into a complete metric space (Y, d_2) such that $f(X)$ is dense in Y with respect to d_2. Then, (Y, d_2) is the completion of (X, d_1).

For example, the completion of $\mathbb{Q}$, with the usual metric, is $\mathbb{R}$, with the usual metric. To see this, define $f : \mathbb{Q} \to \mathbb{R}$ by $f(x) = x$. Then, obviously, f is an isometric embedding. Further, $\mathbb{R}$ is complete and $\overline{\mathbb{Q}} = \mathbb{R}$. This is one way we define the set of all real numbers $\mathbb{R}$. It is simply the completion of $\mathbb{Q}$. The details of early constructions of $\mathbb{R}$ can actually be seen in the proof of Theorem 7.1.20 where the real numbers are technically equivalence classes of Cauchy sequences.

For another example, take the set $\mathbb{R}$ with the metric $d(x, y) = |e^{-x} - e^{-y}|$. We saw earlier in the section that $(\mathbb{R}, d)$ is not complete. Let $Y = \mathbb{R} \cup \infty$, where ∞ can really be anything which isn't an element of $\mathbb{R}$ and define $d_0(x, y)$ on Y by $d_0(x, y) = |e^{-x} - e^{-y}|$, where we define $e^{-\infty} = 0$. Then (Y, d_0) is a complete metric space. If we define $f : \mathbb{R} \to Y$ by $f(x) = x$, then f is an obvious isometric embedding and $\overline{\mathbb{R}} = Y$. Hence, $Y = \mathbb{R} \cup \{\infty\}$ is the completion of $(\mathbb{R}, d)$.

7.2 Homeomorphic Metric Spaces

We have already seen that isometric metric spaces are homeomorphic. We also discussed that metric spaces being isometric is a much stronger condition than them being homeomorphic. In this section, we will first investigate weaker conditions for metric spaces to be homeomorphic. We will then use this new knowledge to investigate when two different metrics, defined on the same set, generate the same topology.

Definition 7.2.1. Let (X, d_1) and (Y, d_2) be two metric spaces and suppose $f : (X, d_1) \to (Y, d_2)$. We say the function f is **Lipschitz** if there exists $K > 0$ such that

$$d_2(f(x_1), f(x_2)) \leq K d_1 d(x_1, x_2) \quad \text{, for all } x_1, x_2 \in X.$$

Exercise 7.2.2. Let (X, d_1) and (Y, d_2) be two metric spaces and suppose $f : (X, d_1) \to (Y, d_2)$ is Lipschitz. Prove f is continuous.

Example 7.2.3. Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5x$, where $\mathbb{R}$ has the usual metric d. Then, for $x_1, x_2 \in \mathbb{R}$,

$$d(f(x_1), f(x_2)) = |f(x_1) - f(x_2)| = |5x_1 - 5x_2| = 5|x_1 - x_2| = 5d(x_1, x_2)$$

and so f is Lipschitz. In fact, any differentiable function $f : (\mathbb{R}, d) \to (\mathbb{R}, d)$ whose derivative is bounded (that is, $f'(\mathbb{R})$ is bounded) is a Lipschitz function. To see why, simply recall the Mean Value Theorem from your calculus class which states that if f is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Written differently,

$$(f(b) - f(a)) = f'(c)(b - a).$$

Taking absolute values, we then have

$$|f(b) - f(a)| = |f'(c)||b - a|.$$

Thus, if $f : (\mathbb{R}, d) \to (\mathbb{R}, d)$ is differentiable, let $x_1, x_2 \in \mathbb{R}$. Without loss of generality, suppose $x_1 < x_2$. Apply the Mean Value Theorem to the interval $[x_1, x_2]$ to obtain that

$$|f(x_2) - f(x_1)| = |f'(c)||x_2 - x_1|.$$

If the derivative is bounded, then there exists $K > 0$ such that $|f'(x)| \leq K$ for all $x \in \mathbb{R}$. Thus, we have that

$$d(f(x_1), f(x_2)) = |f(x_2) - f(x_1)| = |f'(c)||x_2 - x_1| \leq K|x_2 - x_1| = Kd(x_1, x_2).$$

Hence, the function f is Lipschitz.

A simple result, which will get us headed in the right direction, is the following:

Proposition 7.2.4. *Let (X, d_1) and (Y, d_2) be metric spaces and let $f : (X, d_1) \to (Y, d_2)$ be a bijection. If f and f^{-1} are Lipschitz, then (X, d_1) is homeomorphic to (Y, d_2).*

Proof. The proof follows immediately from the fact that Lipschitz functions are continuous. Hence, f is a continuous bijection whose inverse is also continuous. Hence, f is a homeomorphism. □

Definition 7.2.5. Let (X, d_1) and (Y, d_2) be metric spaces. We say a function $f : (X, d_1) \to (Y, d_2)$ is **bi-Lipschitz** if there exists $k, K \in (0, \infty)$ such that

$$k d_1(x_1, x_2) \leq d_2(f(x_1), f(x_2)) \leq K d_1(x_1, x_2)\,, \quad \text{for all } x_1, x_2 \in X.$$

In the literature, we often see a slightly different definition for a bi-Lipschitz function. There, a function is said to be bi-Lipschitz if there exists $K_0 > 0$ such that

$$\frac{1}{K_0} d_1(x_1, x_2) \leq d_2(f(x_1), f(x_2)) \leq K_0 d_1(x_1, x_2) \quad \text{,for all } x_1, x_2 \in X.$$

It is not hard to see that the two definitions are the same. Obviously, this definition implies the one we have. For the other direction, simply let $K_0 = \max\{\frac{1}{k}, K\}$.

Isometries are a special case of bi-Lipschitz functions where we replace the inequalities above with equalities, and where $k = K = 1$.

Exercise 7.2.6. Let (X, d_1) and (Y, d_2) be metric spaces and suppose $f : (X, d_1) \to (Y, d_2)$. Prove that if f is bi-Lipschitz, then f is injective.

This leads us to the following theorem.

Theorem 7.2.7. *Let (X, d_1) and (Y, d_2) be metric spaces and suppose $f : (X, d_1) \to (Y, d_2)$ is bi-Lipschitz. Then (X, d_1) and $(f(X), d_2|_{f(X)\times f(X)})$ are homeomorphic.*

Proof. Suppose $f : (X, d_1) \to (Y, d_2)$ is bi-Lipschitz. By Exercise 7.2.6, we have that f is injective. Also, by Exercise 7.2.2, we have that f is continuous. Thus, $f : (X, d_1) \to (f(X), d_2|_{f(X)\times f(X)})$ is a continuous bijection. Hence, $f^{-1} : (f(X), d_2|_{f(X)\times f(X)}) \to (X, d_1)$ is a well-defined bijection. It suffices to show f^{-1} is continuous. We will do so by showing f^{-1} is Lipschitz. Let $f(x_1), f(x_2) \in f(X)$. Since f is bi-Lipschitz, there exists $k > 0$ such that $kd_1(x_1, x_2) \leq d_2(f(x_1), f(x_2))$. Then,

$$d_1(f^{-1}(f(x_1)), f^{-1}(f(x_2))) = d_1(x_1, x_2) \leq \frac{1}{k} d_2(f(x_1), f(x_2)).$$

Thus, f^{-1} is Lipschitz and therefore continuous. Hence, (X, d_1) and $(f(X), d_2|_{f(X)\times f(X)})$ are homeomorphic. □

The above theorem tells us that if we have a bi-Lipschitz function f from a metric space (X, d_1) to a metric space (Y, d_2), then (X, d_1) is homeomorphic to a subspace of (Y, d_2).

Corollary 7.2.8. *Let (X, d_1) and (Y, d_2) be metric spaces and suppose $f : (X, d_1) \to (Y, d_2)$ is bi-Lipschitz and surjective. Then (X, d_1) and (Y, d_2) are homeomorphic.*

Proof. This follows immediately from Theorem 7.2.7 since $f(X) = Y$. □

We now have one way to show that two metric spaces, while they might not be isometric, have the same topological structure. We can apply this now to the case when we have two different metrics defined on the same set. First, let us start with a definition which will make the discussion easier.

Definition 7.2.9. Let X be a set and let d_1 and d_2 be two metrics on X. Let τ_1 be the topology on X generated by d_1 and let τ_2 be the topology on X generated by d_2. We say d_1 and d_2 are **equivalent metrics** on X if $\tau_1 = \tau_2$.

That is, two metrics defined on the same set X are equivalent if they generate the same topology on X.

Let X be a set and let d_1 and d_2 be metrics on X. If d_1 and d_2 are equivalent, then it's obvious $(X, d_1) \cong (X, d_2)$ since the identity map is a homeomorphism from (X, d_1) to (X, d_2). In fact, the identity map being a homeomorphism also implies the metrics are equivalent, as the next exercise investigates.

Exercise 7.2.10. Let X be a set and let d_1 and d_2 be metrics on X. Prove that d_1 and d_2 are equivalent if and only if the identity map $f : (X, d_1) \to (X, d_2)$, defined by $f(x) = x$, for all $x \in X$, is a homeomorphism.

We can also show that if a metric space is homeomorphic to a topological space, then the topological space is metrizable.

Exercise 7.2.11. Let (X, d) be a metric space and let (Y, τ) be a topological space. Prove that if (X, d) is homeomorphic to (Y, τ) then (Y, τ) is metrizable. *Hint: Let f be a homeomorphism from X to Y and define $d_0 : Y \times Y \to \mathbb{R}$ by $d_0(y_1, y_2) = d(f^{-1}(y_1), f^{-1}(y_2))$. First, prove d_0 is a metric on Y. Then show that a set is open with respect to d_0 if and only if it is an element of τ).*

The next theorem gives a convenient way to think about equivalent metrics, especially with specific examples.

Theorem 7.2.12. *Let X be a set and let d_1 and d_2 be metrics on X. Then d_1 and d_2 are equivalent if and only if, for any point $x \in X$ and any $r > 0$, there exists $r_1, r_2 > 0$ such that*

$$B_{d_1}(x, r_1) \subseteq B_{d_2}(x, r) \quad \textit{and} \quad B_{d_2}(x, r_2) \subseteq B_{d_1}(x, r).$$

Proof. Let X be a set and let d_1 and d_2 be metrics on X. Let τ_1 be the topology on X generated by d_1 and let τ_2 be the topology on X generated by d_2.

Suppose d_1 and d_2 are equivalent. That is, $\tau_1 = \tau_2$. Let $x \in X$ and let $r > 0$. Since $B_{d_2}(x, r)$ is an open neighborhood of x in τ_2 and $\tau_2 = \tau_1$, we have that $B_{d_2}(x, r)$ is an open neighborhood of x in τ_1. Since the open balls with respect to d_1 form a base for τ_1, there exists $r_1 > 0$ such that $B_{d_1}(x, r_1) \subseteq B_{d_2}(x, r)$. Similarly, since $B_{d_1}(x, r)$ is an open neighborhood of x in τ_2, there exists $r_2 > 0$ such that $B_{d_2}(x, r_2) \subseteq B_{d_1}(x, r)$.

Now, suppose for any point $x \in X$ and any $r > 0$, there exists $r_1, r_2 > 0$ such that

$$B_{d_1}(x, r_1) \subseteq B_{d_2}(x, r) \quad \text{and} \quad B_{d_2}(x, r_2) \subseteq B_{d_1}(x, r).$$

We want to show that $\tau_1 = \tau_2$. To this end, let $B_{d_1}(x, r)$ be a basic open set for τ_1, for some $x \in X$ and $r > 0$. Let $y \in B_{d_1}(x, r)$. Then there exists $r_y > 0$ such that $B_{d_1}(y, r_y) \subseteq B_{d_1}(x, r)$. By our assumption, there exists $s_y > 0$ such that $B_{d_2}(y, s_y) \subseteq B_{d_1}(y, r_y)$. Then,

$$B_{d_1}(x, r) = \bigcup_{y \in B_{d_1}(x,r)} B_{d_2}(y, s_y) \in \tau_2.$$

Thus, $\tau_1 \subseteq \tau_2$. A symmetric argument shows $\tau_2 \subseteq \tau_1$. □

Example 7.2.13. Consider the taxicab metric d_1 on $\mathbb{R}^2$. Recall, for $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, the metric d_1 is defined by

$$d_1((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|.$$

Further recall that the open ball $B_{d_1}(\overline{x}, r)$ is an open square rotated by $\frac{\pi}{2}$ to form a diamond shape. Now, let d_2 be the usual metric on $\mathbb{R}^2$. Thus, its open balls are open circles. Now, for $\overline{x} \in \mathbb{R}^2$ and $r > 0$, we can easily see that we can fit an open d_2-ball centered at $\overline{x}$ inside $B_{d_1}(\overline{x}, r)$. That is, we can draw an open circle, centered at $\overline{x}$, inside the open diamond centered at $\overline{x}$. Similarly, given an open circle centered at $\overline{x}$, we can easily draw an open diamond centered at $\overline{x}$ inside the open circle. Thus d_1 and d_2 are equivalent.

Next, we will look at another way to determine whether or not two metrics are equivalent. While it only provides sufficient conditions for two metrics to be equivalent, it is often used in applications.

Theorem 7.2.14. *Let X be a set and let d_1 and d_2 be metrics on X. If the identity function $f : (X, d_1) \to (X, d_2)$ is bi-Lipschitz, then d_1 and d_2 are equivalent.*

Proof. Let the identity map $f : (X, d_1) \to (X, d_2)$ be bi-Lipschitz. By Exercise 7.2.10, it suffices to prove f is a homeomorphism. Obviously, f is a bijection. Further, since f is bi-Lipschitz, we have that f and f^{-1} are Lipschitz. Thus, by Exercise 7.2.2, the functions f and f^{-1} are continuous. Thus, f is a homeomorphism. □

The following corollary simply combines the definition for bi-Lipschitz with the above theorem but it has the advantage of avoiding some terminology and it is often how we show two metrics are equivalent.

Corollary 7.2.15. *Let X be a set and let d_1 and d_2 be metrics on X. If there exists $k, K > 0$ such that*

$$kd_1(x_1, x_2) \leq d_2(x_1, x_2) \leq Kd_1(x_1, x_2)\,, \quad \text{for all } x_1, x_2 \in X,$$

then d_1 and d_2 are equivalent.

Proof. Follows immediately from the definition of bi-Lipschitz and Theorem 7.2.14. □

7.3 Baire Category Theorem

The Baire Category Theorem has many important applications in analysis. One way to formulate the statement of the theorem is that, in a complete metric space, the countable intersection of open dense sets must also be dense. The key word in this statement is "open." For example, $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are both dense in $\mathbb{R}$, which is complete with the usual metric, but $\mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q}) = \emptyset$, which is obviously not dense in $\mathbb{R}$. But, $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are not open in $\mathbb{R}$ so this is not a counterexample to the theorem.

The reason the theorem is called The Baire "Category" Theorem is because of how Baire originally formulated the statement. Let (X, d) be a metric space. We say a subset A of X is **nowhere dense** in X if the interior of its closure is empty. That is, A is nowhere dense if $(\overline{A})^o = \emptyset$. We say a subset of X is of the **first category** if it can be written as a countable union of nowhere dense sets. A subset of X is said to be of the **second category** if it is not of the first category. With this language, The Baire Category Theorem can be stated as follows: In a complete metric space, every nonempty open subset is of the second category. The reader is asked to prove the equivalence of the two formulations of the theorem in the following exercise.

Exercise 7.3.1. Prove that the two formulations of The Baire Category Theorem given above are equivalent. That is, prove the countable intersection of open dense sets is dense if and only if every nonempty open subset is of the second category. *Hint: First prove that if A is nowhere dense, then A^c is dense. Also, a nowhere dense set A is not necessarily closed but $\overline{A}$ is certainly closed. The remainder of the proof relies solely on set operation identities.*

It should be mentioned, the categories discussed by Baire have nothing to do with categories as they are defined in the field of mathematics called Category Theory.

Before proving Baire's theorem, we first need a definition and to prove a different theorem called The Cantor Intersection Theorem.

Definition 7.3.2. Let (X, d) be a metric space. For any subset A of X, we define the **diameter** of A by

$$diam(A) = \sup\{d(x, y) | \, x, y \in A\}$$

if A is bounded and $diam(A) = \infty$ if A is not bounded.

A quick exercise is in order to prove that the definition of the diameter of a set is well-defined.

Exercise 7.3.3. Let (X, d) be a metric space and let A be a subset of X. Then $diam(A) < \infty$ if and only if A is bounded.

We are now ready to prove The Cantor Intersection Theorem.

Theorem 7.3.4. (Cantor Intersection Theorem) *Let (X, d) be a complete metric space and let $(C_n)_{n=1}^{\infty}$ be a sequence of nonempty closed subsets of X such that $C_k \supseteq C_{k+1}$, for all $k \in \mathbb{Z}_+$. If $diam(C_n) \to 0$, then $\cap_{n=1}^{\infty} C_n = \{x\}$, for some $x \in X$.*

Proof. Since each C_n is nonempty, let $x_n \in C_n$, for all $n \in \mathbb{Z}_+$. We first want to show $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence. Let $\epsilon > 0$. Since $diam(C_n) \to 0$, there exists $N_0 \in \mathbb{Z}_+$ such that, for all $n \geq N_0$, we have $diam(C_n) < \epsilon$. Let $N = N_0$ and let $n, m \geq N$. Since $C_n \subseteq C_N$ and $C_m \subseteq C_N$, we have that $x_n, x_m \in C_N$, thus, $d(x_n, x_m) < diam(C_N) < \epsilon$. Hence, $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence.

Since (X, d) is complete, there exists $x \in X$ such that $x_n \to x$. Fix $k \in \mathbb{Z}_+$. Then $(x_n)_{n=k}^{\infty}$ is a sequence in C_k which also converges to x. Since C_k is closed, we have that $x \in C_k$. Thus, we have that $x \in C_k$ for all $k \in \mathbb{Z}_+$ and so $x \in \cap_{n=1}^{\infty} C_n$.

It is left to show there exists no $y \in \cap_{n=1}^{\infty}$, where $y \neq x$. Suppose so. Then $d(x, y) > 0$. Since $diam(C_n) \to 0$, there exists $N \in \mathbb{Z}_+$ such that $diam(C_N) < d(x, y)$. But, $x, y \in C_N$ so $d(x, y) \leq diam(C_N)$. This is a contradiction. Thus, $\cap_{n=1}^{\infty} C_n = \{x\}$. □

We are now ready to prove The Baire Category Theorem.

Theorem 7.3.5. (Baire Category Theorem) *Let (X, d) be a complete metric space. If $(U_n)_{n=1}^{\infty}$ is a sequence of open dense subsets of X, then $\cap_{n=1}^{\infty} U_n$ is dense in X.*

Proof. Let (X, d) be a complete metric space and let $(U_n)_{n=1}^{\infty}$ be a sequence of open desne subsets of X. Let W be a nonempty open set in X. It suffices to show $W \cap (\cap_{n=1}^{\infty} U_n) \neq \emptyset$, by Exercise **??**.

Since U_1 is dense in X, we have that $W \cap U_1 \neq \emptyset$. Further, $W \cap U_1$ is open, so there exists $x_1 \in X$ and $0 < r_1 < \frac{1}{2}$ such that $\overline{B}_d(x_1, r_1) \subseteq W \cap U_1$.

Now, we will construct sequences $(x_n)_{n=1}^{\infty}$ and $(r_n)_{n=1}^{\infty}$ inductively in the following way: Suppose we have constructed x_k and r_k for all $k = 2, \ldots, n$ such that

$$\overline{B}_d(x_k, r_k) \subseteq B_d(x_{k-1}, r_{k-1}) \cap U_k \quad \text{and} \quad 0 < r_k < \frac{1}{2^k}$$

and, for $k = 1$, $\overline{B}_d(x_1, r_1) \subseteq W \cap U_1$ and $0 < r_1 < \frac{1}{2}$. Consider $B_d(x_n, r_n) \cap U_{n+1}$. It is nonempty, so there exists $x_{n+1} \in B_d(x_n, r_n) \cap U_{n+1}$. Further, it is open, so there exists $0 < r_{n+1} < \frac{1}{2^{n+1}}$ such that $\overline{B}_d(x_{n+1}, r_{n+1}) \subseteq B_d(x_n, r_n) \cap U_{n+1}$. Thus, we have constructed sequences $(x_n)_{n=1}^{\infty}$ and $(r_n)_{n=1}^{\infty}$ such that

(i) $\overline{B}_d(x_{n+1}, r_{n+1}) \subseteq B_d(x_n, r_n) \cap U_{n+1}$ for all $n \in \mathbb{Z}_+$, and $\overline{B}_d(x_1, r_1) \subseteq W \cap U_!$, and

(ii) $0 < r_n < \frac{1}{2^n}$, for all $n \in \mathbb{Z}_+$.

Thus, we have a sequence of closed sets $(\overline{B}_d(x_n, r_n))_{n=1}^{\infty}$ such that $r_n \to 0$. Hence, $diam(\overline{B}_d(x_n, r_n)) \leq r_n \to 0$. Further, for all $n \in \mathbb{Z}_+$, we have

$$\overline{B}_d(x_{n+1}, r_{n+1}) \subseteq B_d(x_n, r_n) \subseteq \overline{B}_d(x_n, r_n).$$

So, by The Cantor Intersection Theorem, $\cap_{n=1}^{\infty} \overline{B}_d(x_n, r_n) = \{x\}$, for some $x \in X$. Hence, for all $n > 1$, we have that $x \in \overline{B}_d(x_n, r_n) \subseteq \overline{B}_d(x_{n-1}, r_{n-1}) \cap U_n$, and so $x \in U_n$ for all $n > 1$. Further, $x \in \overline{B}_d(x_1, r_1) \subseteq W \cap U_1$ and so we also have that $x \in W$ and $x \in U_1$. In all, we now have that $x \in W$ and $x \in U_n$ for all $n \in \mathbb{Z}_+$. Thus, $x \in W \cap (\cap_{n=1}^{\infty} U_n)$ and therefore $\cap_{n=1}^{\infty} U_n$ is dense in X. □

As we mentioned, there are many applications of the Baire Category Theorem and we will see some important ones in the next chapter. One interesting application is the following exercise.

Exercise 7.3.6. Prove that $\mathbb{R}^2$, with the usual metric, cannot be covered by a countable number of straight lines.

We can generalize the previous exercise to $\mathbb{R}^n$. Specifically, we can use the Baire Category Theorem to show that $\mathbb{R}^3$ cannot be covered by a countable union of planes or we cannot cover $\mathbb{R}^n$ by a countable union of "planes" (translated $n-1$-dimensional subspaces).

We can also use The Baire Category Theorem to show that the set of all continuous functions on some closed interval $[a, b]$ which are differentiable for at least one point in $[a, b]$ are of the first category in the set of all continuous functions on $[a, b]$ (we can define a metric on the set of all continuous functions on $[a, b]$ which makes it a complete metric space). Thus, "most" continuous functions on $[a, b]$ are nowhere differentiable (not differentiable at any point in $[a, b]$).

7.4 Metrics Defined on Products

Given sets X_i, where $i \in I$, for some index set I, when and how can we define a metric on $\prod_{i\in I} X_i$? Recall from Chapter 5 that $\mathbb{R}^{\mathbb{R}}$ with the product topology is nonmetrizable even though $\mathbb{R}$ with the usual metric is a metric space. Thus, in general, we can't expect the product of metric spaces to have a metric which induces the product topology. In fact, we have the following theorem.

Theorem 7.4.1. *Let (X, τ_i) be a topological space, for all i in some index set I. The set $\prod_{i\in I} X_i$, with the product topology, is metrizable if and only if each (X_i, τ_i) is metrizable and X_i is a singleton for all but countably many $i \in I$.*

Proof. Let $X = \prod_{i \in I} X_i$.

We will start with the forward direction. Since each X_j is homeomorphic to a subspace of X, specifically, $X_j \cong \prod_{i \in I} Y_i$, where $Y_i = \{x_i\}$, for some $x_i \in X_i$, for $i \neq j$, and $Y_j = X_j$, and X is metrizable, we have that X_j is metrizable by Exercise 7.2.11. Further, since X is metrizable, it must be first countable and so there can be, at most, countably many nonsingleton factors in the product.

For the backwards direction, we can simply disregard the singleton factors and assume $I = \mathbb{Z}_+$ since, if X_{i_k}, for $k \in \mathbb{Z}_+$, denote the nonsingleton factors, then $\prod_{k \in \mathbb{Z}_+} X_{i_k} \cong \prod_{i \in I} X_i$. Thus, let's just assume $I = \mathbb{Z}_+$. Let $k \in \mathbb{Z}_+$. Then (X_k, τ_k) is metrizable, so there exists a metric d_k which induces τ_k. Define $\rho_k : X_k \times X_k \to \mathbb{R}$ by $\rho_k(x, y) = \min\{1, d_k(x, y)\}$. A following exercise asks the reader to show ρ_k is a metric which is equivalent to d_k. Thus, ρ_k induces τ_k as well. The advantage of the metric ρ_k is that it has the added property that $\rho_k(x, y) \leq 1$ for all $x, y \in X_k$. Now, for $\overline{x}, \overline{y} \in X$, define

$$d(\overline{x}, \overline{y}) = \sum_{k=1}^{\infty} \frac{\rho_k(x_k, y_k)}{2^k}.$$

It is fairly straightforward to check that d defines a metric on X. What is left to show is that d induces the product topology on X. To this end, let $\overline{x} \in X$ and let U be a basic open neighborhood of $\overline{x}$ with respect to the product topology. Then there exists $n \in \mathbb{Z}_+$, $k_1, \ldots, k_n \in \mathbb{Z}_+$, and $\epsilon_1, \ldots, \epsilon_n > 0$ such that

$$U = \pi_{k_1}^{-1}(B_{\rho_{k_1}}(x_{k_1}, \epsilon_1)) \cap \pi_{k_2}^{-1}(B_{\rho_{k_2}}(x_{k_2}, \epsilon_k)) \cap \cdots \cap \pi_{k_n}^{-1}(B_{\rho_{k_n}}(x_{k_n}, \epsilon_n)).$$

Let $\epsilon = \min\{\frac{\epsilon_1}{2^{k_1}}, \frac{\epsilon_2}{2^{k_2}}, \ldots, \frac{\epsilon_n}{2^{k_n}}\}$. We want to show that $B_d(\overline{x}, \epsilon) \subseteq U$. Let $\overline{y} \in B_d(\overline{x}, \epsilon)$. Then $d(\overline{x}, \overline{y}) < \epsilon$ and so, for $j = 1, \ldots, n$, we have that

$$\frac{\rho_{k_j}(x_{k_j}, y_{k_j})}{2^{k_j}} \leq d(\overline{x}, \overline{y}) < \epsilon \leq \frac{\epsilon_j}{2^{k_j}}$$

and so $\rho_{k_j}(x_{k_j}, y_{k_j}) < \epsilon_j$. Hence, $\overline{y} \in U$. This shows that product topology is weaker than the topology induced by d.

On the otherhand, given $\overline{x} \in X$, a basic open neighborhood of $\overline{x}$ with respect to the topology induced by d is of the form $B_d(\overline{x}, \epsilon)$, for some $\epsilon > 0$. Pick $N \in \mathbb{Z}_+$ such that $\sum_{k=N+1}^{\infty} \frac{1}{2^k} < \frac{\epsilon}{2}$ and let

$$U = \pi_1^{-1}(B_{\rho_1}(x_1, \frac{\epsilon}{2N})) \cap \pi_2^{-1}(B_{\rho_2}(x_2, \frac{\epsilon}{2N})) \cap \cdots \cap \pi_N^{-1}(B_{\rho_N}(x_N, \frac{\epsilon}{2N})).$$

Clearly, U is open with respect to the product topology and $\overline{x} \in U$. We claim that $U \subseteq B_d(\overline{x}, \epsilon)$. Let $\overline{y} \in U$. Then $\rho_k(x_k, y_k) < \frac{\epsilon}{2N}$, for all $k = 1, 2, \ldots, N$. Then,

$$\begin{aligned} d(\overline{x}, \overline{y}) &= \sum_{k=1}^{N} \frac{\rho_k(x_k, y_k)}{2^k} + \sum_{k=N+1}^{\infty} \frac{\rho_k(x_k, y_k)}{2^k} \\ &< \frac{\epsilon}{2N} \sum_{k=1}^{N} \frac{1}{2N} + \sum_{k=N+1}^{\infty} \frac{1}{2^k} \\ &< \frac{\epsilon}{2N} N + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

So, $\overline{y} \in B_d(\overline{x}, \epsilon)$. Hence, we also have the topology induced by d is weaker than the product topology. Therefore, d induces the product topology on X and the proof is complete. □

Exercise 7.4.2. Let (X, d) be a metric space. Define $\rho : X \times X \to \mathbb{R}$ by $\rho(x, y) = \min\{1, d(x, y)\}$ for all $x, y \in X$. First, prove ρ is a metric on X. Then, show that ρ is equivalent to d.

For example, the above theorem tells us that $\mathbb{R}^{\aleph_0}$, with the product topology, is metrizable. It also tells us that for any metric spaces, if there are only countably many of them, then the product of the metric spaces, with the product topology, is metrizable. In particular, if there are only finitely many metric spaces, then their product, with the product topology, is metrizable.

What if, though, given a product of metric spaces, we are willing to have metrics defined on the product, or subsets of the product, which induce different topologies other than the product topology? There are many reasons for wanting to do this. For example, suppose we have an uncountable product of metric spaces and we are willing to have a topology other than the product topology in order for the product space to be metrizable. Or, for example, even though there is a metric on $\mathbb{R}^{\aleph_0}$ which induces the product topology, perhaps we are willing to have a different metric that, while it doesn't induce the product topology, has nice geometric properties which are present in applications. In fact, we can take both scenarios a step further. Suppose we have a product of metrizable spaces and a metric which reflects properties showing up in applications. Unfortunately, the metric is not defined on the entire product space. In this scenario, it is sometimes acceptable to simply define the metric on a subset of the product space where the metric is defined.

We could certainly investigate all of these questions through the lense of metric spaces. For most of them though, the modern context to view them from is that of topological vector spaces. The next chapter introduces the reader to topological vector spaces and investigates some of the above questions from a different perspective.

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