MATH 422-Introduction to Topology

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In this chapter, we investigate some of the important properties of topological spaces. Specifically, we will discuss, what are called the separation axioms, as well as compact spaces, locally compact spaces, and connected spaces. We will also look at some of the consequences of topological spaces having these properties.

6.1 Separation Axioms

Some of the topologies discussed in Chapter 3, while having pedagogical uses, are otherwise quite useless in applications. Take for example the indiscrete topology and the discrete topology which are so exclusive and inclusive, respectively, that they render the topological structure virtually meaningless. What we would like to find is some kind of "happy medium" where we include enough sets to make the topology useful and interesting without overdoing it. It turns out that the way to do this is to construct topologies rich enough to distinguish between different objects (whether they be points or sets) in the space.

Definition 6.1.1. We say a topological space (X, τ) is a T_0**-space**, or say τ is T_0, if for every distinct $x_1, x_2 \in X$, there exists $O \in \tau$ such that O contains x_1 and not x_2 or O contains x_2 and not x_1.

Example 6.1.2. (i) The standard example of a T_0-space is $\mathbb{R}$ with the right ray topology. Given any two points $a, b \in \mathbb{R}$, we can find an open set that contains one and not the other. Indeed, if $a < b$, then (a, ∞) is an open set which contains b but not a. Notice though, and this is important, there does not exist an open set which contains a and not b. This is not required of a T_0-space.

(ii) If X has more than one element and the indiscrete topology τ, then (X, τ) is not a T_0-space.

Definition 6.1.3. We say a topological space (X, τ) is a T_1**-space**, or say τ is T_1, if for every distinct $x_1, x_2 \in X$ there exists $O_1 \in \tau$ such that $x_1 \in O_1$ and $x_2 \notin O_1$, and there exists $O_2 \in \tau$ such that $x_2 \in O_2$ and $x_1 \notin O_2$.

Obviously, every T_1-space is T_0.

Example 6.1.4. (i) If $X = \{1, 2\}$ and $\tau = \{\emptyset, \{1\}, \{1, 2\}\}$, then (X, τ) is a T_0-space but not a T_1-space since, given two points in X (obviously, 1 and 2), we can find an open set which contains 1 and not 2 but we cannot find an open set which contains 2 and not 1.

(ii) The right ray topology on $\mathbb{R}$ is another example of a topology which is T_0 but not T_1, as we have already discussed how, given $a, b \in \mathbb{R}$, where $a < b$, we can find an open set containing b and not a, but we cannot find an open set containing a and not b.

(iii) If we equip $\mathbb{R}$ with the finite complement topology τ, then $(\mathbb{R}, \tau)$ is a T_1-space. The proof of this is part of a later exercise.

One characterization of a T_1-space is given in the following theorem.

Theorem 6.1.5. *A topological space is a T_1-space if and only if every singleton is closed.*

Proof. First, supppose (X, τ) is a T_1-space. Let $x \in X$. Then, for every $y \in X \setminus \{x\}$, there exists $O_y \in \tau$ such that $y \in O_y$ and $x \notin O_y$. Then $\{x\}^c = \cup_{y \in X \setminus \{x\}} O_y \in \tau$. Hence, $\{x\}$ is closed.

For the other direction, suppose all singletons in X are closed. Let x, y be distinct elements of X. Since $\{x\}$ is closed, the set $\{x\}^c$ is an open neighborhood of y that does not contain x. Similarly, since $\{y\}$ is closed, the set $\{y\}^c$ is an open neighborhood of x that does not contain y. Thus, (X, τ) is a T_1-space. $\square$

Definition 6.1.6. A topological space (X, τ) is called a T_2**-space**, or **Hausdorff space**, if for every distinct $x_1, x_2 \in X$, there exists $O_1, O_2 \in \tau$ such that $x_1 \in O_1$, $x_2 \in O_2$, and $O_1 \cap O_2 = \emptyset$. In this case, we also say τ is T_2 or Hausdorff.

Obviously, every T_2-space is a T_1-space. Also, it is clear that every metric space is a T_2-space. Nevertheless, it will be given as a proceeding theorem because of its importance.

Theorem 6.1.7. *Let (X, d) be a metric space. Then the topology generated by d is Hausdorff.*

Proof. Indeed, if (X, d) is a metric space and x and y are distince elements of X, let $r = \frac{1}{2}d(x, y)$. Then $B_d(x, r)$ and $B_d(y, r)$ are disjoint open neighborhoods of x and y, respectively. □

Exercise 6.1.8. Consider $\mathbb{R}$ with the finite complement topology τ. Prove τ is T_1 but not T_2.

Analysts almost exclusively deal with Hausdorff topologies for one very important reason, as the next theorem illustrates.

Theorem 6.1.9. *Let (X, τ) be a Hausdorff topological space. Then the limit of any net in X is unique. That is, if $(x_\lambda)_{\lambda \in \Lambda}$ is a net in X and $x_\lambda \xrightarrow{\lambda} x_1$ and $x_\lambda \xrightarrow{\lambda} x_2$, for some $x_1, x_2 \in X$, then $x_1 = x_2$.*

Proof. Let (X, τ) be Hausdorff and suppose $(x_\lambda)_{\lambda \in \Lambda}$ is a net in X such that $x_\lambda \to x_1$ and $x_\lambda \to x_2$, for some $x_1, x_2 \in X$. Suppose $x_1 \neq x_2$. Then there exists $U_1, U_2 \in \tau$ such that $x_1 \in U_1$, $x_2 \in U_2$, and $U_1 \cap U_2 = \emptyset$. Since $x_\lambda \to x_1$, there exists $\lambda_1 \in \Lambda$ such that, for all $\lambda \geq \lambda_1$, we have that $x_\lambda \in U_1$. Similarly, since $x_\lambda \to x_2$, there exists $\lambda_2 \in \Lambda$ such that, for all $\lambda \geq \lambda_2$, we have that $x_\lambda \in U_2$. Then, pick $\lambda \in \Lambda$ where $\lambda \geq \lambda_1$ and $\lambda \geq \lambda_2$ and we have that $x_\lambda \in U_1 \cap U_2 = \emptyset$, which is a contradiction. Therefore, it must be the case that $x_1 = x_2$. □

Recall that sequences are just a special type of net, so the above theorem tells us that, in a Hausdorff topological space, sequences have unique limits.

Since weak topologies are often used in analysis, and we have just seen why it is important for topologies in analysis to be Hausdorff, we would like to have conditions which imply a weak topology is Hausdorff. Our next theorem provides such conditions.

Theorem 6.1.10. *Let X be a set and let (X_i, τ_i) be a Hausdorff topological space, for all $i \in I$. Let $\mathcal{F} = \{f_i : X \to X_i | \, i \in I\}$ and let τ be the weak topology on X generated by $\mathcal{F}$. If $\mathcal{F}$ separates points, then τ is Hausdorff.*

Proof. Let x_1 and x_2 be distinct points in X. Since $\mathcal{F}$ separates points, there exists $i \in I$ such that $f_i(x_1) \neq f_i(x_2$. Since X_i is Hausdorff, there exists $U_1, U_2 \in \tau_i$ such that $f_i(x_1) \in U_1$, $f_i(x_2) \in U_2$, and $U_1 \cap U_2 = \emptyset$. Then $x_1 \in f_i^{-1}(U_1) \in \tau$, $x_2 \in f_i^{-1}(U_2) \in \tau$, and $f_i^{-1}(U_1) \cap f_i^{-1}(U_2) = \emptyset$. Hence, τ is Hausdorff. □

Exercise 6.1.11. (i) Let (X, τ) be a Hausdorff topological space and let $A \subseteq X$. Prove that A with the subspace topology is a Hausdorff space.

(ii) Let (X_i, τ_i) be a Hausdorff topological space for all $i \in I$. Prove that $\prod_{i \in I} X_i$ with the product topology is a Hausdorff space.

Exercise 6.1.12. Let (X, τ) and (Y, σ) be homeomorphic topological spaces. Prove (X, τ) is a Hausdorff space if and only if (Y, σ) is a Hausdorff space.

We will now give the remaining two separation axioms although we will not discuss them in detail.

Definition 6.1.13. We say a topological space (X, τ) is **regular** if, whenever C is a closed subset of X and $x \notin C$, there exists $O_1, O_2 \in \tau$ such that $C \subseteq O_1$, $x \in O_2$, and $O_1 \cap O_2 = \emptyset$.

As we saw in Theorem 6.1.5, singletons are not all closed unless the topological space is a T_1-space, thus, regular spaces are not necessarily Hausdorff. If we make the further requirement that a regular space be T_1, then we get the next separation axiom.

Definition 6.1.14. If a topological space is a regular T_1-space, then we say it is a T_3**-space**.

If a regular topological space is a T_1-space, so that singletons are closed, then it is obviously Hausdorff.

The last separation axiom is given in the following definition.

Definition 6.1.15. A topological space (X, τ) is **normal** if, given disjoint closed subsets A_1 and A_2 of X, there exist disjoint $O_1, O_2 \in \tau$ such that $A_1 \subseteq O_1$ and $A_2 \subseteq O_2$. If (X, τ) is also a T_1-space, then we say (X, τ) is a T_4**-space**.

Clearly, a T_4-space is a T_3-space.

6.2 Compact Spaces

Definition 6.2.1. Let (X, τ) be a topological space. An **open cover** of X is a collection $\mathcal{C} \subseteq \tau$ such that $X = \cup_{O \in \mathcal{C}} O$. Given an open cover $\mathcal{C}$, a **finite subcover** is a finite subset $\mathcal{C}_0$ of $\mathcal{C}$ such that $X = \cup_{O \in \mathcal{C}_0} O$.

Definition 6.2.2. We say a topological space (X, τ) is **compact** if every open cover of X has a finite subcover.

Example 6.2.3. (i) The set $\mathbb{R}$ with the usual topology is not compact since $\mathcal{C} = \{(-n, n) \mid n \in \mathbb{Z}_+\}$ is an open cover of $\mathbb{R}$ which has no finite subcover.

(ii) Let (X, τ) be a topological space where X is finite. Then (X, τ) must be compact since every open covering would be finite.

(iii) Consider $X = \{\frac{1}{n} | n \in \mathbb{Z}_+\}$ as a subspace of $\mathbb{R}$ with the usual topology. Then X is compact. Indeed, let $\mathcal{C}$ be an open covering of X. Then there exists $O \in \mathcal{C}$ which contains 0. Thus, we can find $\epsilon > 0$ such that $(-\epsilon, \epsilon) \cap X \subseteq O$. So, for some $N \in \mathbb{Z}_+$, we have that for all $n \geq N$, the point $\frac{1}{n} \in O$. Now, for $k = 1, \ldots, N-1$, there exists $O_k \in \mathcal{C}$ such that $\frac{1}{k} \in O_k$ and so $\mathcal{C}' = \{O_1, \ldots, O_{N-1}, O\}$ is a finite subcover of X.

(iv) The interval $[0, 1)$, as a subspace of $\mathbb{R}$ with the usual topology, is not compact since $\mathcal{C} = \{[0, 1 - \frac{1}{n}) | n \in \mathbb{Z}_+\}$ is an open cover of $[0, 1)$ with no finite subcover.

Next, we want to prove that every interval of the form $[a, b]$, as a subspace of $\mathbb{R}$ with the usual topology, is compact. We first need a fact about the real numbers.

Definition 6.2.4. Let A be a subset of $\mathbb{R}$. We say $b \in \mathbb{R}$ is an **upper bound** for A if $a \leq b$ for all $a \in A$. We say $b \in \mathbb{R}$ is the **least upper bound** of A if b is an upper bound for A and if b_0 is another upper bound for A, then $b \leq b_0$. In this case, we also refer to b as the **supremum** of A, and write $b = \sup A$. Similarly, we say $c \in \mathbb{R}$ is a **lower bound** for A if $c \leq a$ for all $a \in A$. We say $c \in \mathbb{R}$ is the **greatest lower bound** of A if c is a lower bound for A and if c_0 is another lower bound for A, then $c_0 \leq c$. In this case, we also refer to c as the **infimum** of A, and write $c = \inf A$.

Fact: Every nonempty subset of $\mathbb{R}$ which has an upper bound, has a supremum (that is, has a least upper bound) and every nonempty subset of $\mathbb{R}$ which has a lower bound, has an infimum (that is, a greatest lower bound).

Whether the above fact is a theorem or an axiom depends on how we define the real numbers. Most modern analysis textbooks have it as an axiom and so it is a defining characteristic of the real numbers. That is, the real numbers are the smallest set containing the rational numbers where the above fact is true.

Theorem 6.2.5. *Let $a, b \in \mathbb{R}$, where $a < b$. The interval $[a, b]$, as a subspace of $\mathbb{R}$ with the usual topology, is compact.*

Proof. A closed interval $[a, b]$ in $\mathbb{R}$ with the subspace topology is compact. Let $\mathcal{C}$ be an open cover of $[a, b]$. Let D be the set of all points d in $[a, b]$ such that a finite subcollection of $\mathcal{C}$ covers $[a, d]$. Clearly, $a \in D$ so D is nonempty. Since b is an upper bound for D, the set D has a least upper bound. Let $d = \sup D$. Since there exists $O \in \mathcal{C}$ such that $a \in O$, we can find $\delta > 0$ such that $[a, \delta) \subseteq O$ so, for any $x \in [a, \delta)$, we have that $x \in D$. Thus, $d \neq a$.

If $d = b$ then we are done so suppose $d < b$. Since $\mathcal{C}$ is an open cover of $[a, b]$ and $d \in [a, b]$, there exists $O \in \mathcal{C}$ such that $d \in O$. Since $a < d < b$, there exists $\epsilon > 0$ such that $(d - \epsilon, d + \epsilon) \subseteq O$. Since d is the least upper bound of D, there exists $c \in (d - \epsilon, d)$ such that $c \in D$ (if not, $d - \epsilon$ would be an upper bound for D which is less than d). Thus, there exists a finite subcover $\mathcal{C}_0$ of $[a, c]$. But then, for any $e \in (d, d + \epsilon)$, the collection $\mathcal{C}_0 \cup \{O\}$ is a finite subcover of $[a, e]$ which implies that $e \in D$. This contradicts the fact that d is an upper bound for D. Thus, we must have that $d = b$ and so there exists a finite subcover for $[a, b]$ and so $[a, b]$ is compact. □

In a first course of analysis, we learn that a subset K of $\mathbb{R}$ or $\mathbb{R}^n$ is compact if and only if every sequence in K has a subsequence which converges to some $k \in K$. This is not true for a general topological space. We call such spaces with this property *sequentially compact*.

Definition 6.2.6. A topological space (X, τ) is called **sequentially compact** if every sequence in X has a convergent subsequence.

For a general topological space, we have to replace sequences in the above definition with nets. This result will be given in Theorem 6.2.9. First, we need to look at another way to characterize compact spaces for general topological spaces.

Definition 6.2.7. We say a collection of sets $\mathcal{C} = \{C_i | \, i \in I\}$ has the **finite intersection property** if for every finite subset $F \subseteq I$, we have that $\cap_{i \in F} C_i \neq \emptyset$.

Theorem 6.2.8. *A topological space (X, τ) is compact if and only if every family of closed sets with the finite intersection property has nonempty intersection.*

Proof. First, suppose (X,τ) is compact. Let $\mathcal{C} = \{C_i |\, i \in I\}$ be a family of closed sets with the finite intersection property. Suppose $\cap_{i\in I} C_i = \emptyset$. Then $X = \cup_{i\in I} C_i^c$ is an open cover of X. Since X is compact, there exists a finite set $F \subseteq I$ such that $X = \cup_{i\in F} C_i^c$, but then $\cap_{i\in F} C_i = \emptyset$ contradicting the fact that $\mathcal{C}$ has the finite intersection property.

For the other direction, suppose every family of closed sets with the finite intersection property has nonempty intersection. Let $\mathcal{O} = \{O_i |\, i \in I\}$ be an open covering of X. If (X,τ) is not compact, then, for every finite $F \subseteq I$, we have that $X \neq \cup_{i\in F} O_i$ and so $\cap_{i\in F} O_i^c \neq \emptyset$. Thus, $\mathcal{C} = \{O_i^c |\, i \in I\}$ has the finite intersection property. By our assumption, we then have that $\cap_{i\in I} O_i^c \neq \emptyset$ and so $\cup_{i\in I} O_i \neq X$, contradicting the fact that $\mathcal{O}$ is an open cover of X. Therefore, we must have that (X,τ) is compact. □

Theorem 6.2.9. *A topological space (X,τ) is compact if and only if every net in X has a convergent subnet.*

Proof. First, suppose (X,τ) is compact. Let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in X. To show $(x_\lambda)_{\lambda\in\Lambda}$ has a convergent subnet, it suffices to show, by Theorem **??**, that $(x_\lambda)_{\lambda\in\Lambda}$ has a cluster point. Let $C_\lambda = \overline{\{x_\beta |\, \beta \geq \lambda\}}$, for all $\lambda \in \Lambda$. Let $\mathcal{C} = \{C_\lambda |\, \lambda \in \Lambda\}$.

We now want to prove the family $\mathcal{C}$ has the finite intersection property. Let $F = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. Pick $\lambda_0 \geq \lambda_k$, for all $k = 1, 2, \ldots, n$. Then $x_{\lambda_0} \in C_{\lambda_k}$, for all $k = 1, 2, \ldots, n$ so $\cap_{k=1}^n C_{\lambda_k} \neq \emptyset$. Then, since X is compact, we have $\cap_{\lambda\in\Lambda} C_\lambda \neq \emptyset$. Let $x \in \cap_{\lambda\in\Lambda} C_\lambda$. We want to show x is a cluster point. Let U be an open neighborhood of x. Let $\lambda_0 \in \Lambda$. Since $x \in \cap_{\lambda\in\Lambda} C_\lambda$, we have that $x \in C_{\lambda_0} = \overline{\{x_\beta |\, \beta \geq \lambda_0\}}$, we have that $U \cap \{x_\beta |\, \beta \geq \lambda_0\} \neq \emptyset$. So, there exists $\lambda \geq \lambda_0$ such that $x_\lambda \in U$. Thus, x is a cluster point.

For the other direction, suppose $\mathcal{C} = \{C_i |\, i \in I\}$ is a family of closed sets with the finite intersection property. Let $\mathcal{P}$ be the set of all finite subsets of I with an order defined by $F_1 \leq F_2$ if and only if $F_1 \subseteq F_2$. For $F \in \mathcal{P}$, we have that $\cap_{i\in F} C_i \neq \emptyset$, so let $x_F \in \cap_{i\in F} C_i$. Thus, we have a net $(x_F)_{F\in\mathcal{P}}$. By assumption, the net $(x_F)_{F\in\mathcal{P}}$

has a cluster point x. We now want to prove that $x \in C_i$ for all $i \in I$. Let $i_0 \in I$. Let U be an open neighborhood of x. Then, for $F_0 = \{i_0\}$, there exists $F \geq F_0$ such that $x_F \in U$. Since $F \geq F_0$, we have that $i_0 \in F$. Also, we have $x_F \in \cap_{i \in F} C_i$, so $x_F \in C_{i_0}$. Hence, $U \cap C_{i_0} \neq \emptyset$. Thus, $x \in \overline{C_{i_0}} = C_{i_0}$. Therefore, $x \in C_i$, for all $i \in I$ and so $x \in \cap_{i \in I} C_i$. □

While it was mentioned in the proof above that, because of Theorem **??**, Theorem 6.2.9 is equivalent to the statement that a space is compact if and only if every net has a cluster point, it is worth stating as a following corollary.

Corollary 6.2.10. *A topological space (X, τ) is compact if and only if every net in X has a cluster point.*

Exercise 6.2.11. (i) Let (X, τ) be a compact space. Prove that every closed set A in X is compact (when A is equipped with the subspace topology).

(ii) Let (X, τ) be a Hausdorff space. If A is a compact subset of X, then A is closed.

Exercise 6.2.12. Prove that the continuous image of a compact space is compact. That is, suppose (X, τ) and (Y, σ) are topological spaces and $f : (X, \tau) \to (Y, \sigma)$ is continuous. Prove that if (X, τ) is a compact space, then $f(X)$ is a compact subspace of (Y, σ). Conclude that if (X, τ) is homeomorphic to (Y, σ) then (X, τ) is compact if and only if (Y, σ) is compact.

Our next goal is to prove Tychonoff's Theorem which states that a product of topological spaces, with the product topology, is compact if and only if each factor in the product is compact. We first need a remark and a few definitions. The proof provided in these notes follows a proof given by Chernoff in 1992 in a paper titled, "A simple proof of Tychonoff's Theorem via nets."

Remark 6.2.13. If (X_i, τ_i) is a topological space, for all $i \in I$ and we let $X = \prod_{i \in I} X_i$ with the product topology, then any basic open set N in X is of the form

$$N = \pi_{i_1}^{-1}(U_{i_1}) \cap \pi_{i_2}^{-1}(U_{i_2}) \cap \cdots \cap \pi_{i_n}^{-1}(U_{i_n}) \tag{6.1}$$

for some $n \in \mathbb{Z}_+$, where $U_{i_j} \in \tau_{i_j}$ for all $j = 1, \ldots, n$. If we let $F = \{i_1, i_2, \ldots, i_n\}$, then we will denote the basic open set N in Equation (6.1) by $N = N\{U_j | \, j \in F\}$.

Definition 6.2.14. Let (X_i, τ_i) be a topological space, for all $i \in I$ and let $X = \prod_{i \in I} X_i$ with the product topology. If $J \subseteq I$, then we say $g \in \prod_{i \in J} X_i$ is a **partially defined member** of X. Let $(f_\lambda)_{\lambda \in \Lambda}$ be a net in X. Suppose g, with domain J, is a partially defined member of X. We say g is a **partial cluster point** of $(f_\lambda)_{\lambda \in \Lambda}$ if, given $\lambda_0 \in \Lambda$, for every finite subset $F \subseteq J$ and every basic open neighborhood $N\{U_j | \, j \in F\}$ of g in $\prod_{i \in J} X_i$, there exists $\lambda \in \Lambda$, where $\lambda \geq \lambda_0$ such that, for all $j \in F$, we have that $f_\lambda(j) \in U_j$. Or, equivalently, g is a partial cluster point for $(f_\lambda)_{\lambda \in \Lambda}$ if g is a cluster point in $\prod_{i \in J} X_i$ for the net $(f_\lambda|_J)_{\lambda \in \Lambda}$.

Theorem 6.2.15. (Tychonoff's Theorem) *Let (X_i, τ_i) be a topological space for all $i \in I$. Then $X = \prod_{i \in I} X_i$ with the product topology is compact if and only if (X_i, τ_i) is compact for every $i \in I$.*

Proof. First, suppost X is compact. Let $i \in I$. Since the projection map $\pi_i : X \to X_i$ is continuous, by Exercise 6.2.12, we have that $\pi_i(X) = X_i$ is compact.

Now, for the other direction, suppose (X_i, τ_i) is compact, for all $i \in I$. We want to show that every net in X has a subnet which converges to an element of X. Since a net has a convergent subnet to a point g if and only if g is a cluster point for the net, it suffices to show that every net in X has a cluster point. To this end, let $(f_\lambda)_{\lambda \in \Lambda}$ be a net in X. Our goal is to show that there exists a partial cluster point g, with domain J, such that $J = I$. Then g is a cluster point for $(f_\lambda)_{\lambda \in \Lambda}$.

To this end, let $\mathcal{P}$ be the set of all partial cluster points of $(f_\lambda)_{\lambda \in \Lambda}$. The set $\mathcal{P}$ is nonempty since the empty function is an element of $\mathcal{P}$. Let $g_1, g_2 \in \mathcal{P}$, where J_1 is the domain of g_1 and J_2 is the domain of g_2. Define a partial ordering on $\mathcal{P}$ by $g_1 \leq g_2$ if and only if $J_1 \subseteq J_2$ and $g_1(j) = g_2(j)$, for all $j \in J_1$. We now want to show $\mathcal{P}$ has a maximal element by using Zorn's Lemma. Let $\mathcal{C} = \{g_\alpha | \, \alpha \in A\}$ be a chain in $\mathcal{P}$. Define $g_0 = \cup_{\alpha \in A} g_\alpha$ (where we think of g_α as a subset of $J_\alpha \times (\cup_{i \in I} X_i)$). Since $\mathcal{C}$ is totally ordered, it is easy to see that g_0 is a partially defined member of X with domain $\cup_{\alpha \in A} J_\alpha$ and that, since each g_α is a partial cluster point for $(f_\lambda)_{\lambda \in \Lambda}$, the function g_0 is also a partial cluster point for $(f_\lambda)_{\lambda \in \Lambda}$. Hence, $g_0 \in \mathcal{P}$. Clearly, g_0 is an upper bound for $\mathcal{C}$. Thus, by Zorn's Lemma, $\mathcal{P}$ contains a maximal member g.

We now wish to show that if J is the domain of g, then $J = I$. Suppose not. Let $k \in I \setminus J$. Since g is a cluster point in $\prod_{i \in J} X_i$ for $(f_\lambda|_J)_{\lambda \in \Lambda}$, there exists a subnet $(f_{\lambda_\mu}|_J)_{\mu \in M}$ such that $f_{\lambda_\mu}|_J \xrightarrow{\mu} g$. Since X_k is compact and $(f_{\lambda_\mu}(k))_{\mu \in M}$ is a net in X_k, $(f_{\lambda_\mu}(k))_{\mu \in M}$ has a cluster point p in X_k. Define a function h on $J \cup \{k\}$ by $h(j) = g(j)$, for all $j \in J$ and $h(k) = p$. Then h is a partial cluster point of $(f_\lambda)_{\lambda \in \Lambda}$, so $h \in \mathcal{P}$, and $h > g$, which contradicts the maximality of g in $\mathcal{P}$. Hence, we must have that $J = I$ and so g is a cluster point of $(f_\lambda)_{\lambda \in \Lambda}$. Therefore, X is compact. □

Tychonoff's Theorem has many important applications. As Stephen Willard wrote in his textbook *General Topology*, "the theorem just proved (Tychonoff's Theorem) can lay good claim to being the most important theorem in general (nongeometric) topology." One application of Tychonoff's Theorem is the Heine-Borel Theorem. We first need to define bounded sets in metric spaces.

Definition 6.2.16. Let (X, d) be a metric space and let $A \subseteq X$. We say the set A is **bounded** if there exists $x \in X$ and $r > 0$ such that $A \subseteq B_d(x, r)$. We say a sequence $(x_n)_{n=1}^{\infty}$ is **bounded** if the set $\{x_n | \, n \in \mathbb{Z}_+\}$ is bounded.

Exercise 6.2.17. Let (X, d) be a metric space and let $(x_n)_{n=1}^{\infty}$ be a sequence in X. If $(x_n)_{n=1}^{\infty}$ converges, then $(x_n)_{n=1}^{\infty}$ is bounded.

Theorem 6.2.18. (Heine-Borel Theorem) *A subset of* $\mathbb{R}^n$ *with the usual topology is compact if and only if it is closed and bounded.*

Proof. Let $A \subseteq \mathbb{R}^n$.

First, suppose A is compact. By Exercise 6.2.11, since $\mathbb{R}^n$ is Hausdorff and A is compact, we have that A is closed. To show A is bounded, let $\mathcal{C} = \{B_d(x,1) \cap A | \, x \in A\}$, where d is the usual metric on $\mathbb{R}^n$. Then $\mathcal{C}$ is an open cover of A so there exists a finite subcover. That is, there exists $n \in \mathbb{Z}_+$ and $x_1, x_2, \ldots, x_n \in A$ such that $A \subseteq \cup_{k=1}^n B_d(x,1)$. Let $r = \max\{|x_1|, |x_2|, \ldots, |x_n|\} + 1$. Then $A \subseteq B_d(0,r)$ and so A is bounded.

Now, suppose A is closed and bounded. Since A is bounded, $A \subseteq B(0,M)$, for some $M > 0$. Hence, $A \subseteq [-M,M]^n$ which is compact by Tychonoff's Theorem. Thus, A is a closed subspace of a compact space and so, by Exercise 6.2.11, we have that A is compact. □

Since $\mathbb{C}^n$ with the usual topology is homeomorphic to $\mathbb{R}^{2n}$, the Heine-Borel Theorem also tells us that subsets of $\mathbb{C}^n$ are compact if and only if they are closed and bounded.

One statement whose proof is easily extracted from our proof of the Heine-Borel Theorem is that if we have a compact subset of a metric space, then it is closed and bounded.

Corollary 6.2.19. *Let* (X,d) *be a metric space and let* $A \subseteq X$. *If* A*, with the subspace topology, is compact, then* A *is closed and bounded.*

Proof. Since metric spaces are Hausdorff, by Exercise 6.2.11, we have that A is closed. To show A is bounded, consider $\mathcal{C} = \{B_d(x,1) \cap A | \, x \in A\}$. Clearly, $\mathcal{C}$ is an open cover of A. Since A is compact there must exist $n \in \mathbb{Z}_+$ and $x_1, \ldots, x_n \in A$ such that $A = \cup_{k=1}^n B_d(x_k, 1)$. Let $r = \max\{d(x_1, x_k) | \, k \in \mathbb{Z}_n\} + 1$. Then $A \subseteq B_d(x_1, r)$ and so A is bounded. □

In an analysis course, one important fact about a continuous functon $f : A \to \mathbb{R}$, where $A \subseteq \mathbb{R}^n$, is that if A is compact, then the function f is bounded. We can now prove this statement in much more generality as the next exercise shows.

Exercise 6.2.20. Let (X, τ) be a compact topological space and let (Y, d) be a metric space. If $f : (X, \tau) \to (Y, d)$ is continuous, then $f(X)$ is bounded.

We can actually quite a bit more in the above exercise if the metric space (Y, d) is $\mathbb{R}$ with the usual metric.

Theorem 6.2.21. (Extreme Value Theorem) *Let (X, τ) be a compact topological space and let $f : (X, \tau) \to (\mathbb{R}, \sigma)$ be continuous, where σ is the usual topology on $\mathbb{R}$. Then there exist $x_1, x_2 \in X$ such that $f(x_1) \leq f(x) \leq f(x_2)$ for all $x \in X$.*

Proof. Since (X, τ) is compact and f is continuous, we have that $f(X)$ is a compact subset of $\mathbb{R}$. By the Heine-Borel Theorem, we then have that $f(X)$ is closed and bounded. Since $f(X)$ is bounded, we know $s = \sup f(X)$ exists. If s is not a limit point of $f(X)$, then there exists $\epsilon > 0$ such that $(f(X) \cap (s - \epsilon, s + \epsilon)) \setminus \{s\} = \emptyset$. But then, any $y \in (s - \epsilon, s)$ would be an upper bound for $f(X)$ and would be less than s, contradicting the fact that s is the *least* upper bound. Thus, we must have that s is a limit point of $f(X)$. Then, since $f(X)$ is closed, we must have that $s \in f(X)$. Thus, there exists $x_2 \in X$ such that $f(x_2) = s$ and so $f(x_2) \geq f(x)$ for all $x \in X$. A similar argument shows there exists $x_1 \in X$ such that $f(x_1) = \inf f(X)$ and so $f(x_1) \leq f(x)$ for all $x \in X$. □

It should be stressed that while compact implies closed and bounded in a metric space, the converse is not necessarily true. That is, in a general metric space, closed and bounded does not imply compact. This illustrates the significance of the Heine-Borel Theorem. If our metric space is $\mathbb{R}^n$ with the usual metric, then the converse is true.

It is worth noting that the concepts of sequential compactness and compactness are not weaker or stronger assumptions than the other. There are examples of compact spaces which are not sequentially compact and there are examples of sequentially compact spaces which are not compact. In many instances, checking that a space is sequentially compact tends to be easier than checking whether every open covering has a finite subcovering so it would be nice to have an easy way to know when the two types of compactness are equivalent. The next theorem tells us that a metric space is sequentially compact if and only if it is compact. We first need two lemmas.

Lemma 6.2.22. (Lebesgue Number Lemma) *Let (X, d) be a sequentially compact metric space. If $\mathcal{O}$ is an open cover of X, then there exists $\delta > 0$ such that, for each $x \in X$, there exists $O \in \mathcal{O}$ such that $B_d(x, \delta) \subseteq O$.*

Proof. Suppose not. Let $\mathcal{O}$ be an open covering for X. Then for all $\delta > 0$, there exists $x \in X$ such that $B_d(x, \delta) \nsubseteq O$, for all $O \in \mathcal{O}$. In particular, for all $n \in Z_+$, there exists $x_n \in X$ such that $B_d(x_n, \frac{1}{n}) \nsubseteq O$, for all $O \in \mathcal{O}$. Thus, we have defined a sequence $(x_n)_{n=1}^{\infty}$ in X. Since (X, d) is sequentially compact, there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ and $x \in X$ such that $x_{n_k} \to x$. Since $\mathcal{O}$ is an open cover of X, there exists $O \in \mathcal{O}$ such that $x \in O$. Since O is open, there exists $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq O$. Since $x_{n_k} \to x$, there exists $K_0 \in \mathbb{Z}_+$ such that, for all $k \geq K_0$, we have $d(x_{n_k}, x) < \frac{\epsilon}{2}$. Pick $k \geq K_0$ so that $\frac{1}{n_k} < \frac{\epsilon}{2}$.

Claim: $B_d(x_{n_k}, \frac{1}{n_k}) \subseteq B_d(x, \epsilon)$.

Indeed, for $x_0 \in B_d(x_{n_k}, \frac{1}{n_k})$, we have that $d(x_0, x_{n_k}) < \frac{1}{n_k} < \frac{\epsilon}{2}$. So

$$d(x_0, x) \leq d(x_0, x_{n_k}) + d(x_{n_k}, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, $B_d(x_{n_k}, \frac{1}{n_k}) \subseteq B_d(x, \epsilon) \subseteq O$ and $O \in \mathcal{O}$, so this is a contradiction. □

Given an open covering, a number δ satisfying the conditions in the lemma above is called a *Lebesgue number* for the covering which is the reason for the lemma's title.

Lemma 6.2.23. *If (X, d) is a sequentially compact metric space, then, for all $\epsilon > 0$, there exists $n \in Z_+$ and $x_1, x_2, \ldots, x_n \in X$ such that*

$$X = \bigcup_{k=1}^{n} B_d(x_k, \epsilon).$$

Proof. We will proceed with a proof by contrapositive so suppose there exists $\epsilon > 0$ such that, for any $n \in \mathbb{Z}_+$ and all $x_1, x_2, \ldots, x_n \in X$, we have that $X \neq \cup_{k=1}^{n} B_d(x_k, \epsilon)$.

Let $x_1 \in X$. By our assumption, $X \neq B_d(x_1, \epsilon)$ and so pick $x_2 \in X$ such that $x_2 \notin B_d(x_1, \epsilon)$. Again, by our assumption, $X \neq B_d(x_1, \epsilon) \cup B_d(x_2, \epsilon)$ so pick $x_3 \in X$ such that $x_3 \notin B_d(x_1, \epsilon) \cup B_d(x_2, \epsilon)$. For $n \geq 2$, suppose we have defined $x_1, x_2, \ldots, x_{n-1}$. By our assumption, $X \neq \cup_{k=1}^{n-1} B_d(x_k, \epsilon)$ so let $x_n \in X$ such that $x_n \notin \cup_{k=1}^{n-1} B_d(x_k, \epsilon)$. Thus, using induction, we have defined a sequence $(x_n)_{n=1}^{\infty}$ such that $x_n \notin \cup_{k=1}^{n-1} B_d(x_k, \epsilon)$, for all $n \in \mathbb{Z}_+$.

We now claim that $(x_n)_{n=1}^{\infty}$ has no convergent subsequence. Indeed, for $n, m \in Z_+$ such that $n > m$, we have that $x_n \notin B_d(x_m, \epsilon)$ and so $d(x_n, x_m) \geq \epsilon$. Thus, the distance between any two distinct terms in the sequence must be at least ϵ and so it cannot have a convergent subsequence. □

Theorem 6.2.24. *Let (X, d) be a metric space. Then (X, d) is sequentially compact if and only if it is compact.*

Proof. Suppose (X, d) is sequentially compact. Let $\mathcal{O}$ be an open cover of X. From Lemma 6.2.22, there exists $\delta > 0$ such that, for all $x \in X$, there exists $O \in \mathcal{O}$ such that $B_d(x, \delta) \subseteq O$. By Lemma 6.2.23, there exists $n \in Z_+$ and $x_1, x_2, \ldots, x_n \in X$ such that

$$X = \bigcup_{k=1}^{n} B_d(x_k, \delta).$$

But, for each $k = 1, 2, \ldots, n$, there exists $O_k \in \mathcal{O}$ such that $B_d(x_k, \delta) \subseteq O_k$. Thus, $X = \cup_{k=1}^n O_k$ and so $\{O_1, O_2, \ldots, O_n\}$ is a finite subcover of $\mathcal{O}$ and so (X, d) is compact.

We will prove the backwards direction by proving the contrapositive so suppose (X, d) is not sequentially compact. Then there exists a sequence $(x_n)_{n=1}^\infty$ in X which has no convergent subsequence. Let $z \in X$. Then there exists $\delta_z > 0$ so that, at most, only finitely many terms from $(x_n)_{n=1}^\infty$ are contained in $B_d(z, \delta_z)$ (If not, then for all $\epsilon > 0$, we would have infinitely many terms from the sequence contained in $B_d(z, \epsilon)$ thus allowing us to construct a subsequence which converges to z). Then, we have

$$X = \bigcup_{z \in X} B_d(z, \delta_z)$$

and so $\mathcal{O} = \{B_d(z, \delta_z) |\, z \in X\}$ is an open cover of X. But, for any $n \in Z_+$ and $z_1, z_2, \ldots, z_n \in X$, we have

$$X \neq \bigcup_{k=1}^n B_d(z_k, \delta_{z_k})$$

since each $B_d(z_k, \delta_{z_k})$ contains only finitely many terms from the sequence $(x_n)_{n=1}^\infty$ and so there must exist terms from the sequence which are not contained in $\cup_{k=1}^n B_d(z_k, \delta_{z_k})$. Hence, $\mathcal{O}$ does not have a finite subcover and so (X, d) is not compact. □

A consequence of the above theorem as well as the Heine-Borel Theorem is the Bolzano-Weierstrass Theorem. We first need a quick lemma.

Lemma 6.2.25. *Let (X, d) be a metric space and let $A \subseteq X$. If A is bounded, then $\overline{A}$ is bounded.*

Proof. Let A be a bounded subset of X. Then there exists $x \in X$ and $r > 0$ such that $A \subseteq B_d(x, r)$. Then we have that $A \subseteq B_d(x, r) \subseteq \overline{B_d}(x, r)$ and, since $\overline{B_d}(x, r)$ is closed, $\overline{A} \subseteq \overline{B_d}(x,r)$. Thus,

$$\overline{A} \subseteq \overline{B_d}(x,r) \subseteq B_d(x,r+1)$$

and so $\overline{A}$ is bounded. □

Theorem 6.2.26. (Bolzano-Weierstrass Theorem) *Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

Proof. Let $(x_n)_{n=1}^{\infty}$ be a bounded sequence in $\mathbb{R}^n$. Let $A = \{x_n | \, n \in \mathbb{Z}_+\}$. Then A is bounded. By Lemma 6.2.25, we have that $\overline{A}$ is bounded. Thus, $\overline{A}$ is a closed and bounded subset of $\mathbb{R}^n$. By the Heine-Borel Theorem, $\overline{A}$ is compact. Since $\mathbb{R}^n$ is a metric space where compactness implies sequential compactness, we have that every sequence in $\overline{A}$ has a convergent subsequence. In particular, $(x_n)_{n=1}^{\infty}$ is a sequence in $\overline{A}$ and, thus, it has a convergent subsequence. □

6.3 Locally Compact Spaces

Compactness is such a useful property that we often seek to involve it in spaces which are not compact. One way we can do this is to introduce the concept of local compactness. Let's start with the definition.

Definition 6.3.1. A topological space (X, τ) is **locally compact at** x_0, where $x_0 \in X$, if there exists an open neighborhood U of x_0 and a compact subspace K of X such that $U \subseteq K$. We say a topological space (X, τ) is **locally compact** if it is locally compact at x_0 for all $x_0 \in X$.

Example 6.3.2. (i) Every compact set is locally compact. Indeed, if (X, τ) is compact, then for any $x_0 \in X$, the set X is an open neighborhood of x_0 which is contained in the compact set X.

(ii) The set $\mathbb{R}$ with the usual topology is locally compact. Given any $x \in \mathbb{R}$, the point x is an element of the open interval $(x-1, x+1)$ which is contained in the compact set $[x-1, x+1]$. The set $\mathbb{R}$ with the usual topology is not, however, compact.

(iii) For any $n \in Z_+$, the set $\mathbb{R}^n$ with the topology induced by the usual metric d is locally compact. Given $x \in \mathbb{R}^n$, we have that $B_d(x,1) \subseteq \overline{B}_d(x,1)$ and $\overline{B}_d(x,1)$ is compact by the Heine-Borel Theorem, since it is closed and bounded.

(iv) The set $\mathbb{R}^{\mathbb{R}}$ with the product topology is not locally compact. Indeed, for $f \in \mathbb{R}^{\mathbb{R}}$, any open set containing f would contain a base element of the form

$$B = \pi_{x_1}^{-1}((a_1, b_1)) \cap \cdots \cap \pi_{x_n}^{-1}((a_n, b_n))$$

and any compact set containing B would have to contain $\overline{B}$ (since $\mathbb{R}^{\mathbb{R}}$ is Hausdorff, by Exercise 6.2.11, part (ii)). However,

$$\overline{B} = \pi_{x_1}^{-1}([a_1, b_1]) \cap \cdots \cap \pi_{x_n}^{-1}([a_n, b_n])$$

and, by Tychonoff's Theorem, $\overline{B}$ is compact if and only if $\pi_x(\overline{B})$ is compact for all $x \in \mathbb{R}$. But, for $x \neq x_i$ for $i = 1, \ldots, n$, we have that $\pi_x(\overline{B}) = \mathbb{R}$ which is not compact.

Theorem 6.3.3. *Let (X, τ) and (Y, σ) be topological spaces and suppose $f : (X, \tau) \to (Y, \sigma)$ is continuous, open, and onto. If (X, τ) is locally compact, then (Y, σ) is locally compact.*

Proof. Let $y_0 \in Y$. Let O be an open neighborhood of y_0. Since f is onto, there exists $x_0 \in X$ such that $f(x_0) = y_0$ Since f is continuous, we have that $f^{-1}(O)$ is an open neighborhood of x_0. Since (X, τ) is locally compact, there exists an open neighborhood U of x_0 and a compact set K such that $f(U) \subseteq f(K)$. Then, $V = U \cap f^{-1}(O)$ is an open neighborhood of x_0. Further, since f is open, the set

$f(V)$ is an open neighborhood of y_0. Also, $V \subseteq K$ so $f(V) \subseteq f(K)$. Since f is continuous and K is compact, by Exercise 6.2.12 we have that $f(K)$ is compact. Thus, $f(V)$ is an open neighborhood of y_0 and $f(V)$ is contained in the compact set $f(K)$ and so (Y, σ) is locally compact at y_0. This holds for all $y_0 \in Y$ and so (Y, σ) is locally compact. □

Exercise 6.3.4. If (X, τ) is homeomorphic to (Y, σ) prove that (X, τ) is locally compact if and only if (Y, σ) is locally compact.

Theorem 6.3.5. *Let (X_α, τ_α) be a topological space, for all $\alpha \in I$. Then $\prod_{\alpha \in I} X_\alpha$ is locally compact if and only if each (X_α, τ_α) is locally compact and all but finitely many are compact.*

Proof. Suppose $\prod_{\alpha \in I} X_\alpha$ is locally compact. Let $\gamma \in I$. Since the projection map $\pi_\gamma : \prod_{\alpha \in I} X_\alpha \to X_\gamma$ is continuous, open, and onto, by Theorem 6.3.3, we have that (X_γ, τ_γ) is locally compact.

Let $f \in \prod_{\alpha \in I} X_\alpha$. Since $\prod_{\alpha \in I} X_\alpha$ is locally compact, there exists a basic open neighborhood U of f and a compact set K such that $U \subseteq K$. Since U is a basic open set, there exists $n \in \mathbb{Z}_+$, $\alpha_1, \alpha_2, \ldots, \alpha_n \in I$, and $O_{\alpha_k} \in \tau_{\alpha_k}$, for all $k = 1, 2, \ldots, n$, such that

$$U = \pi_{\alpha_1}^{-1}(O_{\alpha_1}) \cap \pi_{\alpha_2}^{-1}(O_{\alpha_2}) \cap \cdots \cap \pi_{\alpha_n}^{1}(O_{\alpha_n}).$$

Let $\gamma \in I$ such that $\gamma \neq \alpha_k$, for all $k = 1, 2, \ldots, n$. Then we have that $\pi_\gamma(K)$ is compact, since π_γ is continuous. And, since $U \subseteq K$, we have that $\pi_\gamma(U) \subseteq \pi_\gamma(K)$. But, since $\gamma \neq \alpha_k$, for all $k = 1, 2, \ldots, n$, we have that $\pi_\gamma(U) = X_\gamma$. Thus, $X_\gamma = \pi_\gamma(K)$ is compact. Thus, the only X_α which are potentially not compact are $X_{\alpha_1}, X_{\alpha_2}, \ldots, X_{\alpha_n}$.

For the other direction, suppose (X_α, τ_α) is locally compact for all $\alpha \in I$ and compact except for $\alpha_1, \alpha_2, \ldots, \alpha_n \in I$, for some $n \in Z_+$. Let $f \in \prod_{\alpha \in I} X_\alpha$. For $k = 1, 2, \ldots, n$, we have that $f(\alpha_k) \in X_{\alpha_k}$. Since X_{α_k} is locally compact, there exists an open neighborhood $O_{\alpha_k} \in \tau_{\alpha_k}$ of $f(\alpha_k)$ and a compact set K_{α_k} in X_{α_k} such that $O_{\alpha_k} \subseteq K_{\alpha_k}$. Then,

$$U = \pi_{\alpha_1}^{-1}(O_{\alpha_1}) \cap \pi_{\alpha_2}^{-1}(O_{\alpha_2}) \cap \cdots \cap \pi_{\alpha_n}^{-1}(O_{\alpha_n})$$

is an open neighborhood of f, and

$$K = \pi_{\alpha_1}^{-1}(K_{\alpha_1}) \cap \pi_{\alpha_2}^{-1}(K_{\alpha_2}) \cap \cdots \cap \pi_{\alpha_n}^{-1}(K_{\alpha_n})$$

is compact by Tychonoff's Theorem (Theorem 6.2.15) since K is the product of compact sets. Also, since $O_{\alpha_k} \subseteq K_{\alpha_k}$ for all $k = 1, 2, \ldots, n$, we have that $U \subseteq K$. Thus, $\prod_{\alpha \in I} X_\alpha$ is locally compact. □

Example 6.3.6. The above theorem is another way to see that $\mathbb{R}^n$ is locally compact, for all $n \in \mathbb{Z}_+$. It is also another way to see that $\mathbb{R}^{\mathbb{R}}$ is not locally compact.

In Chapter 3, the motivation for defining the closure of a set A was that it was the smallest closed set which contained A. Can we do something similar with compact sets? That is, given a set, can we define the smallest compact set which contains it? Let us now make this more precise.

Definition 6.3.7. Let (Y, σ) be a compact Hausdorff space and let X be a proper subspace of Y. If $\overline{X} = Y$, then we say Y is the **compactification** of X.

Example 6.3.8. (i) Consider the interval $[a, b]$ in $\mathbb{R}$ with the usual topology. Since $\overline{(a, b)} = [a, b]$, then $[a, b]$ is the compactification of (a, b).

(ii) More generally, if we consider $\overline{B}(x, r)$ in $\mathbb{R}^n$, for some $x \in \mathbb{R}^n$, then, since $\overline{B(x, r)} = \overline{B}(x, r)$ and $\overline{B}(x, r)$ is closed and bounded, hence compact, we have that $\overline{B}(x, r)$ is the compactification of $B(x, r)$.

Determining which topological spaces have compactifications and finding descriptions of those compactifications is a complicated task. It turns out though, that if a topological space is locally convex and Hausdorff, then, not only does it have a compactification, but the compactification is obtained by simply including one additional point. Furthermore, this property characterizes locally compact Hausdorff spaces as the next theorem illustrates.

Theorem 6.3.9. *Let (X, τ) be a topological space. Then (X, τ) is locally compact and Hausdorff if and only if there exists a topological space (Y, σ) such that*

(i) (X, τ) is a subspace of (Y, σ),

(ii) $Y \setminus X$ is a singleton, and

(iii) (Y, σ) is a compact Hausdorff space.

Further, if (Y, σ) and (Y', σ') are two topological spaces satisfying these conditions, then (Y, σ) and (Y', σ') are homeomorphic and the homeomorphism between them is the identity map on X.

Proof. First, suppose (X, τ) is a locally compact Hausdorff space. Pick any object which is not an element of X and label it ∞. Let $Y = X \cup \{\infty\}$. Now, define

$$\sigma = \tau \cup \{Y \setminus C | C \text{ is a compact subspace of } X\}. \tag{6.2}$$

It is a following exercise to show that σ is a topology on Y. We now want to check that (X, τ) is a subspace of (Y, σ). That is, we want to show $\sigma|_X = \tau$. To this end, let $O \in \tau$. Then $O \subseteq X$ and, by construction, $O \in \sigma$, so $O = O \cap X \in \sigma|_X$. Now, let $O \in \sigma|_X$. Thus, $O = U \cap X$, for some $U \in \sigma$. If $U \in \tau$, then $U \subseteq X$ and so $O = U \in \tau$ so suppose $U = Y \setminus C$, for some compact subspace C of X. Then

$$O = U \cap X = (Y \setminus C) \cap X = X \setminus C \in \tau$$

since C is closed by Exercise 6.2.11. Hence, (X, τ) is a subspace of (Y, σ).

Now, to show (Y, σ) is compact, let $\mathcal{O}$ be an open covering of Y. Then there must exist $U \in \mathcal{O}$ of the form $U = Y \setminus C$, for some compact subspace C of X (these are the only sets in σ which contain ∞). Now, consider $\mathcal{O}_0 = \{O \cap X | O \in \mathcal{O} \setminus \{U\}\}$. Then $\mathcal{O}_0$ is a collection of open sets in (X, τ). Further, if $x \in C$, then $x \in X$, since $C \subseteq X$, and x is contained in one of the members of $\mathcal{O}$. Thus, x is contained in one of the members of $\mathcal{O}_0$. Hence, $\mathcal{O}_0$ is an open covering of C. Since C is compact, there exists $O_1, O_2, \ldots, O_n \in \mathcal{O}$, for some $n \in \mathbb{Z}_+$, such that $C \subseteq \cup_{k=1}^{n} O_k \cap X$. Thus, the collection $\{O_1, O_2, \ldots, O_n, U\}$ is a finite subcover of $\mathcal{O}$, and so Y is compact.

To see that (Y, σ) is Hausdorff, let $z, w \in Y$, where $z \neq w$. If $z, w \in X$, then, since (X, τ) is Hausdorff, there exists $U, V \in \tau$ such that $z \in U$, $w \in V$, and $U \cap V = \emptyset$. Since $\tau \subseteq \sigma$, we have what we need. The other possibility is that $z \in X$ and $w = \infty$. Since (X, τ) is locally compact, there exists $U \in \tau$ and a compact set K such that $z \in U \subseteq K$. Then $z \in U \in \tau \subseteq \sigma$, $w \in Y \setminus K \in \sigma$, and $U \cap (Y \setminus K) = \emptyset$. Thus, (Y, σ) is Hausdorff.

Next, to prove the backward direction, suppose (X, τ) is a subspace of a compact Hausdorff space (Y, σ) (note then that $\sigma|_X = \tau$) where $Y \setminus X = \{\infty\}$ (the set $Y \setminus X$ contains exactly one element which we will label ∞). We just have to show (X, τ) is Hausdorff and locally compact. We know (X, τ) is Hausdorff by Exercise **??** since it is a subspace of a Hausdorff space so we are left to show that (X, τ) is locally compact.

Let $x_0 \in X$. Since (Y, σ) is Hausdorff, let $U, V \in \sigma$ such that $x_0 \in U$, $\infty \in V$, and $U \cap V = \emptyset$. Let $C = Y \setminus V$. Then C is a closed subspace of Y and so compact by Exercise 6.2.11. Note that $\infty \notin C$, since $\infty \in V$, thus $C \subseteq X$. Since $\tau = \sigma|_X$, we have that C is a compact subspace of (X, τ). Further, since $\infty \notin U$ we have that $U \subseteq X$, and, since $U \cap V = \emptyset$, we have that $U \subseteq C$. Thus, we have found $U \in \tau$ and a compact subspace C of X such that $x_0 \in U \subseteq C$. Thus, (X, τ) is locally compact.

Lastly, we have to show that if (X, τ) is locally compact and we find two topological spaces (Y, σ) and (Y', σ') satisfying properties (i)-(iii), then there exists a homeomorphism $h : (Y, \sigma) \to (Y', \sigma')$ such that $h|_X(x) = x$, for all $x \in X$. To this end, suppose we have two such topological spaces. Let ∞ be the sole element of $Y \setminus X$ and let ∞' be the sole element of $Y' \setminus X$. Define $h : Y \to Y'$ by $h(x) = x$, for all $x \in X$, and $h(\infty) = \infty'$. Clearly, h is bijective.

Let $U \in \sigma$. We want to show $h(U) \in \sigma'$.

Case 1: The point $\infty \notin U$. Then $U \subseteq X$ and so $U \in \tau$. Since (X, τ) is a subspace of (Y', σ') and $X \in \sigma'$ (since singletons are closed in Hausdorff spaces), we have that $U \in \sigma'$. Then, since $U \subseteq X$, we have $h(U) = U \in \sigma'$.

Case 2: The point $\infty \in U$. Let $C = Y \setminus U$. Then C is a compact subspace of Y, since C is closed and Y is compact. Further, $\infty \notin C$ so C is a compact subspace of X. Then, we have that C is a compact subspace of Y'. Since Y' is Hausdorff, by Exercise 6.2.11, the set C is closed in Y'. Then, $h(U) = Y' \setminus C \in \sigma'$.

Hence, for all $U \in \sigma$, we have that $h(U) \in \sigma'$. A symmetric argument shows that for all $U \in \sigma'$, we have $h^{-1}(U) \in \sigma$. Thus, h is a homeomorphism and the proof is complete. □

Exercise 6.3.10. Prove that the collection σ defined in Equation (6.2) of Theorem 6.3.9 is a topology.

Thus, locally compact Hausdorff spaces are precisely the spaces constructed when we take a compact Hausdorff space and punch a single hole in them.

Definition 6.3.11. Let (Y, σ) be a topological space and let X be a proper subspace of Y. If Y is the compactification of X and $Y \setminus X$ is a singleton, then we say Y is the **one-point compactification** of X.

Theorem 6.3.9 tells us that the only topological spaces with one point compactifications are locally compact Hausdorff spaces which are not themselves compact. The fact that one-point compactifications are unique up to homeomorphism justifies the language being used in the above definition when we say *the* one-point compactification, rather than *a* one-point compactification. Also note that the proof of Theorem 6.3.9 provides a way for us to construct the one-point compactification of a locally compact Hausdorff space.

Example 6.3.12. (i) If we let $\hat{S}_1$ be the unit circle in $\mathbb{R}^2$ with the north pole deleted (that is, the point $(0, 1)$), then $\hat{S}_1$ is clearly locally compact and Hausdorff and its compactification is S_1.

(ii) We have already seen that $\mathbb{R}$ with the usual topology is locally compact and Hausdorff. Thus it must have a one-point compactification. Indeed, if we define $h : \mathbb{R} \to \hat{S}_1$ (where $\hat{S}_1$ is given in the previous example), by

$$h(t) = \left(\cos\left(2\arctan\left(t\right)\right) - \frac{\pi}{2}, \sin\left(2\arctan\left(t\right)\right) - \frac{\pi}{2}\right)$$

then we see that $\mathbb{R}$ is homeomorphic to $\hat{S}_1$ and thus the one-point compactification of $\mathbb{R}$ is homeomorphic to S_1.

(iii) In a similar fashion, if we let $\hat{S}_2$ be the unit sphere in $\mathbb{R}^3$ with the north pole missing (the point $(0, 0, 1)$), then $\mathbb{R}^2$ with the usual topology is homeomorphic to $\hat{S}_2$ which obviously has S_2 as its compactification. Thus, the compactification of $\mathbb{R}^2$ is homeomorphic to S_2. Further, since we have already seen that $\mathbb{C}$ is homeomorphic to $\mathbb{R}^2$, the one-point compactification of $\mathbb{C}$ is also S_2. We often denote the one-point compactification of $\mathbb{C}$ by $\mathbb{C} \cup \{\infty\}$, where ∞ is obviously the new element included with $\mathbb{C}$ to make it compact, and refer to $\mathbb{C} \cup \{\infty\}$ as the **extended complex plane**.

6.4 Connected Spaces

Definition 6.4.1. Let (X, τ) be a topological space. A **separation** of X is a pair of disjoint nonempty sets $U, V \in \tau$ such that $X = U \cup V$. We say the topological space (X, τ) is **connected** if there does not exist a separation of X. We say a topological space (X, τ) is **disconnected** if it is not connected.

Exercise 6.4.2. Prove that a topological space (X, τ) is disconnected if and only if there exist disjoint closed sets A and B such that $X = A \cup B$.

Example 6.4.3. (i) Obvious examples to build intuition would be to take $X = [1, 2] \cup [3, 4]$ with the usual topology. Then, with the subspace topology, $[1, 2]$ and $[3, 4]$ are both open sets so X is not connected.

(ii) If we instead take $X = [1, 2]$ with the usual topology, then there does not exist a separation of X so X is connected.

Another way to characterize connected spaces is given in the following theorem.

Theorem 6.4.4. *A topological space (X, τ) is connected if and only if the only clopen sets in X are $\emptyset$ and X.*

Proof. Suppose (X, τ) is connected. If there exists a set $A \subseteq X$ which is clopen, where $A \neq \emptyset$ and $A \neq X$, then A is open and A^c is open (since A is closed) and so A and A^c form a separation of X. This is a contradiction and so no such A exists.

Now, suppose the only clopen sets in X are $\emptyset$ and X. If X is disconnected, then there exist disjoint nonempty open sets U and V such that $U \cup V = X$. Then, we have that $U^c = V$ and, since U is open, $V = U^c$ is closed. Thus, V is clopen and $V =\neq emptyset$ and $V \neq X$. This is a contradiction. Thus, no such separation exists. □

Exercise 6.4.5. Prove that the continuous image of a connected space is connected. That is, suppose (X,τ) and (Y,σ) are topological spaces and $f:(X,\tau)\to(Y,\sigma)$ is continuous. If (X,τ) is connected, then $f(X)$ as a subspace of Y, is connected. Conclude that if (X,τ) is homeomorphic to (Y,σ) then (X,τ) is connected if and only if (Y,σ) is connected.

Subspaces of connected spaces are not necessarily connected. For example, $[0,10]$ is connected while $[0,1]\cup[9,10]$ is disconnected. The next few propositions inverstigate situations when we can say something about the connectedness of subspaces.

Proposition 6.4.6. *Let (X,τ) be a topological space and let $Y\subseteq X$. The set Y, with the subspace topology, is disconnected if and only if there exist $A,B\subseteq X$ such that $A\cup B=Y$, $\overline{A}\cap B=\emptyset$, and $A\cap\overline{B}=\emptyset$.*

Proof. Suppose Y, with the subspace topology, is disconnected. Then there exist $U,V\in\tau$ such that $(U\cap Y)\cap(V\cap Y)=\emptyset$ and $Y=(U\cap Y)\cup(V\cap Y)$. Let $A=U\cap Y$ and $B=V\cap Y$. Then $A\cup B=Y$ and $A\cap B=\emptyset$. Since the complement of A in Y is B, which is open as a subspace of Y, A is closed in Y, so $A=\overline{A}^{Y}$. Further, from Theorem **??**, $\overline{A}^{Y}=\overline{A}\cap Y$, so $A=\overline{A}\cap Y$. Thus,

$$\emptyset=A\cap B=(\overline{A}\cap Y)\cap B=\overline{A}\cap(Y\cap B)=\overline{A}\cap B$$

since $B\subseteq Y$. A symmetric argument shows that $A\cap\overline{B}=\emptyset$.

Now, suppose $A,B\subseteq X$, $A\cup B=Y$, $\overline{A}\cap B=\emptyset$, and $A\cap\overline{B}=\emptyset$. Since $A\subseteq\overline{A}$, we have that $A\cap B\subseteq\overline{A}\cap B=\emptyset$. Thus, by Exercise 6.4.2, it suffices to show A and B

are closed subsets of Y with respect to the subspace topology on Y. Well,

$$\begin{aligned}
\overline{A}^Y &= \overline{A} \cap Y && \text{by Theorem ??} \\
&= \overline{A} \cap (A \cup B) && \text{since } A \cup B = Y \\
&= (\overline{A} \cap A) \cup (\overline{A} \cap B) \\
&= A && \text{since } A \subseteq \overline{A} \text{ and } \overline{A} \cap B = \emptyset
\end{aligned}$$

Thus, A is closed as a subset of Y. A symmetric argument shows B is also closed as a subset of Y. Hence, A and B form a separation of $(Y, \tau|_Y)$ and thus $(Y, \tau|_Y)$ is disconnected. □

Example 6.4.7. If we consider $Y = [0,1) \cup (1,2]$ as a subspace of $\mathbb{R}$ with the usual topology, then Y is disconnected. Indeed, if we let $A = [0,1)$ and $B = (1,2]$, then $Y = A \cup B$ while $\overline{A} \cap B = [0,1] \cap (1,2] = \emptyset$ and $A \cap \overline{B} = [0,1) \cap [1,2] = \emptyset$.

Proposition 6.4.8. *Let (X, τ) be a topological space and let A and B form a separation of X. If $(Y, \tau|_Y)$ is a connected subspace of (X, τ), then $Y \subseteq A$ or $Y \subseteq B$.*

Proof. Since $A, B \in \tau$, we have that $A \cap Y, B \cap Y \in \tau|_Y$. Further, $(A \cap Y) \cap (B \cap Y) \subseteq A \cap B = \emptyset$. So, $A \cap Y$ and $B \cap Y$ would form a separation of Y unless either $A \cap Y = \emptyset$ or $B \cap Y = \emptyset$. Since $X = A \cup B$, we then have that either $Y \subseteq B$ or $Y \subseteq A$. □

Proposition 6.4.9. *Let (X, τ) be a topological space and let $Y_i \subseteq X$ for all $i \in I$. If $(Y_i, \tau|_{Y_i})$ is connected, for all $i \in I$, and $\cap_{i \in I} Y_i \neq \emptyset$, then $Y = \cup_{i \in I} Y_i$ is connected with respect to $\tau|_Y$.*

Proof. Let $z \in \cap_{i \in I} Y_i$. Suppose A and B form a separation of Y. Then $z \in A$ or $z \in B$. Without loss of generality, suppose $z \in A$. Let $i \in I$. Since $(Y_i, \tau|_{Y_i})$ is a connected subspace of $(Y, \tau|_Y)$, by Proposition 6.4.8, we have that $Y_i \subseteq A$ or $Y_i \subseteq B$. Since $z \in A$ and $z \in Y_i$, we must have that $Y_i \subseteq A$. This holds for all $i \in I$ and thus, $Y = \cup_{i \in I} Y_i \subseteq A$ which implies $B = \emptyset$. This contradicts the fact that A and B forma separation of Y. Thus, $(Y, \tau|_Y)$ is connected. □

Exercise 6.4.10. Let (X, τ) be a topological space and suppose $(A, \tau|_A)$ is connected. If B is a set such that $A \subseteq B \subseteq \overline{A}$, then $(B, \tau|_B)$ is connected.

Note that a consequence of the last exercise is that the closure of a connected space is also connected. It actually says a bit more. If we have a connected space and we want to include some, but perhaps not all, of the limit points, then we can do so and the space will stay connected.

The next theorem tells us when the finite product of topological spaces is connected.

Theorem 6.4.11. *Let (X_i, τ_i) be a topological space, for all $i \in \mathbb{Z}_n$. Then $X = \prod_{i\in\mathbb{Z}_n} X_i$, with the product (or box) topology is connected if and only if (X_i, τ_i) is connected, for all $i \in I$.*

Proof. Suppose X is connected. Let $i \in \mathbb{Z}_n$. Since $\pi_i : X \to X_i$ is continuous, by Exercise 6.4.5, we have that $\pi_i(X) = X_i$ is connected.

For the other direction, first suppose $n = 2$. Fix $(a, b) \in X_1 \times X_2$ and consider the subspaces $\{a\} \times X_2$ and $X_1 \times \{b\}$. Since $\{a\} \times X_2$ is homeomorphic to X_2 and $X_1 \times \{b\}$ is homeomorphic to X_1, we have that $\{a\} \times X_2$ and $X_1 \times \{b\}$ are connected, by Exercise 6.4.5.

Now, for $y \in X_1$, we also have that $\{y\} \times X_2$ is connected, since it is homeomorphic to X_2. Thus, the set $Z_y = (\{y\} \times X_2) \cup (X_1 \times \{b\})$ is connected since it is the union of connected spaces which have the point (y, b) in common. Then, $X_1 \times X_2 = \cup_{y\in X_1} Z_y$ is connected since it is the union of connected spaces which all have the point (a, b) in common.

A simple proof by induction, building off of the ideas above, will then give us the result we want for an arbitrary $n \in \mathbb{Z}_+$. $\square$

Of course, a natural question to ask at this point would be, "Do we have the above theorem when we have an infinite product of topological spaces?" The answer is "yes," if we equip the product space with the product topology. That is, $X = \prod_{i \in I} X_i$, where I is any index set and X has the product topology, is connected if and only if X_i is connected for all $i \in I$. It is not the case, however, if X has the box topology. It is not difficult to find examples in the literature where each X_i is connected yet the product space X, with the box topology, is not connected.

We now wish to prove that $\mathbb{R}$, with the usual topology, is connected. Recall the fact from an earlier section that every bounded subset of $\mathbb{R}$ has a least upper bound.

Theorem 6.4.12. *The set $\mathbb{R}$, with the usual topology, is connected.*

Proof. First, we want to show $I = [0, 1]$ is connected. To this end, suppose A and B form a separation of I. Then 1 must be an element of one of them. Without loss of generality, suppose $1 \in A$. Since A is open, there exists a set of the form $(c, 1]$ such that $(c, 1] \subseteq A$. Thus, $s = \sup B \neq 1$. Well, $s \in [0, 1]$ so $s \in A$ or $s \in B$. Since A and B are open, whichever set contains s must also contain an open neighborhood of s. But, any open neighborhood containing s would contain elements of B (otherwise s would not be the *least* upper bound for B) and elements of A (otherwise s would not be an upper bound for B at all). Either way, this is a contradiction. Thus, no such separation exists and so $I = [0, 1]$ is connected.

Now, for $n \in \mathbb{Z}_+$, $I_n = [-n, n]$ is connected since it is homeomorphic to I. Then, $\mathbb{R} = \cup_{n \in \mathbb{Z}_+} [-n, n]$ is connected since it is the union of connected spaces which all have the point 0 in common. □

Since $\mathbb{R}$ is connected, we can now use Theorem 6.4.11 to see that $\mathbb{R}^n$ is connected, for all $n \in \mathbb{Z}_+$. Also, since $\mathbb{C}^n$ is homeomorphic to $\mathbb{R}^{2n}$, we also have that $\mathbb{C}^n$ is connected, for all $n \in \mathbb{Z}_+$. Note that we also showed in the proof of the last theorem that the interval $[0, 1]$ is connected and, since $[0, 1]$ is homeomorphic to $[a, b]$ for any $a, b \in \mathbb{R}$ with $a < b$, we also have that $[a, b]$ is connected. This is worth making a corollary.

Corollary 6.4.13. *The interval* $[a, b]$, *where* $a, b \in \mathbb{R}$ *and* $a < b$, *is connected.*

Our last goal for this section is to prove the Intermediate Value Theorem from our calculus classes by proving a theorem which is slightly more general. The Intermediate Value Theorem will then be a corollary.

Theorem 6.4.14. *Let* (X, τ) *be a connected topological space and let* σ *be the usual topology on* $\mathbb{R}$. *Let* $f : (X, \tau) \to (\mathbb{R}, \sigma)$ *be continuous. If* $a, b \in X$ *and* r *is an element of* $\mathbb{R}$ *strictly between* $f(a)$ *and* $f(b)$, *then there exists* $c \in X$ *such that* $f(c) = r$.

Proof. Let $f : (X, \tau) \to (\mathbb{R}, \sigma)$ be continuous, where (X, τ) is connected. Let $a, b \in X$ and suppose r is strictly between $f(a)$ and $f(b)$. Consider the sets $A = f(X) \cap (-\infty, r)$ and $B = f(X) \cap (r, \infty)$. Clearly, A and B are disjoint. Also, one of the sets contains $f(a)$ while the other contains $f(b)$ so neither set is empty. Also, both A and B are open subsets of $f(X)$ with the subspace topology. If there exists no $c \in X$ such that $f(c) = r$, then A and B would form a separation of $f(X)$ but we know, from Exercise 6.4.5, that $f(X)$ is connected. Thus, there must exist $c \in X$ such that $f(c) = r$. □

Corollary 6.4.15. (Intermediate Value Theorem) *Let* σ *be the usual topology on* $\mathbb{R}$ *and let* $a < b$. *Suppose* $f : ([a, b], \sigma|_{[a,b]}) \to (\mathbb{R}, \sigma)$ *is continuous and* $r \in \mathbb{R}$ *is strictly between* $f(a)$ *and* $f(b)$. *Then there exists* $c \in (a, b)$ *such that* $f(c) = r$.

Definition 6.4.16. Let X be a set. We say a function $f : X \to X$ has a **fixed point** if there exists $x \in X$ such that $f(x) = x$.

There are many theorems which prove the existence of fixed points for certain types of functions defined on various types of topological spaces (such theorems are called *fixed-point theorems*). Fixed-point theorems are used in many applications, including image processing. The reader is asked to prove one fixed-point theorem below.

Exercise 6.4.17. Let $f : [a, b] \to [a, b]$ be continuous, where $[a, b]$ is a closed interval in $\mathbb{R}$ with the usual topology. Prove that f has a fixed point. *Hint: Define a function $g(x) = f(x) - x$ on $[a, b]$ and use The Intermediate Value Theorem.*

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