MATH 422-Introduction to Topology

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January 27, 2026

Contents

Chapter 1

Preliminaries

Chapter 2

Bridging the Gap: Metric Spaces to Topological Spaces

Chapter 3

Introduction to Topological Spaces

Chapter 4

New Topologies from Old

Chapter 5

Sequences vs. Nets

Anyone who has taken a course in analysis understands the importance of sequences. For example, they can be used to tell us when sets are open or closed, when sets are compact, or when functions are continuous. In a sense, they are a wonderful tool in the analyst's toolbox. In many regards, sequential convergence encapsilates the topological structure of the real numbers. For example, we can say a set O is open if and only if, for every $x \in O$, if $(x_n)_{n=1}^{\infty}$ is a sequence such that $x_n \to x$, then the terms of the sequence must eventually be in O. We can say a set C is closed if and only if, for every convergent sequence in C, the limit must also be in C. For function continuity, a function $f : X \to Y$ is continuous if and only if, for every sequence $(x_n)_{n=1}^{\infty}$ in X such that $x_n \to x \in X$, we have that $f(x_n) \to f(x)$. Even though we haven't discussed compact sets in a topological space yet, a subset D of $\mathbb{R}$ (or $\mathbb{R}^n$) is compact if every sequence in D has a subsequence which converges to an element of D.

Since sequences are such a helpful tool when studying $\mathbb{R}$ (or even $\mathbb{R}^n$) with the usual topology, it would be nice to know if, and when, we can use them in general topological spaces for the same kinds of tasks. In this chapter, we will see under which conditions we can continue to use sequences in this manner. In the settings where we cannot use sequences, we will have other mathematical objects, called nets, which can be used to achieve similar tasks.

5.1 First Countable Spaces

Definition 5.1.1. Let (X, τ) be a topological space. An **open neighborhood base** for a point $x \in X$, is a collection $\mathcal{B}_x$ of open neighborhoods of x such that, given any open neighborhood U of x, there exists $B \in \mathcal{B}_x$ such that $x \in B \subseteq U$.

Example 5.1.2. (i) If we take $\mathbb{R}$ with the usual topology, then $\mathcal{B}_0 = \{(-\frac{1}{n}, \frac{1}{n}) | \, n \in \mathbb{Z}_+\}$ would be an open neighborhood base for 0.

(ii) If (X, d) is a metric space, and $x_0 \in X$, then $\mathcal{B}_{x_0} = \{B_d(x_0, r) | r > 0\}$ is an open neighborhood base for x_0.

Definition 5.1.3. A topological space (X, τ) is **first countable** if and only if every $x \in X$ has a countable open neighborhood base.

Example 5.1.4. (i) The set $\mathbb{R}$ with the usual topology is first countable since, for every $x \in \mathbb{R}$, the set $\mathcal{B}_x = \{(x - \frac{1}{n}, x + \frac{1}{n}) | n \in \mathbb{Z}_+\}$ is a countable open neighbhorhood base for x.

(ii) More generally, every metric space is first countable. Let (X, d) be a metric space and let $x_0 \in X$. The set $\mathcal{B}_{x_0} = \{B_d(x_0, r) | r > 0 \text{ and } r \in \mathbb{Q}\}$ is a countable open neighborhood base for x_0.

(iii) Consider $\mathbb{R}$ with the finite complement topology τ. Then $(\mathbb{R}, \tau)$ is not first countable.

One of the most consequential properties of a first countable space is given in the next theorem.

Theorem 5.1.5. *If (X, τ) is a first countable topological space and $A \subseteq X$, then $x \in \overline{A}$ if and only if there exists a sequence $(x_n)_{n=1}^{\infty}$ in A such that $x_n \xrightarrow{\tau} x$.*

Proof. Let $x \in \overline{A}$. Since (X, τ) is first countable, there exists a countable open neighborhood base $\mathcal{B}_x = \{B_1, B_2, \dots\}$ of x. Let $U_1 = B_1$ and $U_k = \cap_{i=1}^{k} B_k$ for all $k \geq 2$. Now we have that each U_k is an open neighborhood of x and $U_k \supseteq U_{k+1}$ for all $k \in \mathbb{Z}_+$.

Claim: $U_k \cap A \neq \emptyset$, for all $k \in Z_+$.

If there exists $k \in \mathbb{Z}_+$ such that $U_k \cap A = \emptyset$, then U_k^c is a closed set containing A and, since $x \notin U_k^c$, we have that $x \notin \overline{A}$, which is a contradiction. Thus, $U_k \cap A \neq \emptyset$ for all $k \in \mathbb{Z}_+$.

Let $x_k \in U_k \cap A$ for all $k \in Z_+$. Then we have defined a sequence $(x_n)_{n=1}^{\infty}$ in A. It suffices to prove $x_n \to x$. Let O be an open neighborhood of x. Since $\mathcal{B}_x$ is an open neighborhood base for x, there exists $B_N \in \mathcal{B}_x$ such that $x \in B_N \subseteq O$. Let $n \geq N$. Then

$$x_n \in U_n \subseteq U_N \subseteq B_N \subseteq O.$$

Thus, $x_n \to x$.

For the other direction, suppose there exists a sequence $(x_n)_{n=1}^{\infty}$ in A such that $x_n \to x$. If $x \not{i}n\overline{A}$, then $x \in \overline{A}^c$, which is open. Hence, $\overline{A}^c$ is an open neighborhood of x. Since $x_n \to x$, there exists $N \in \mathbb{Z}_+$ such that, for all $n \geq N$, we have $x_n \in \overline{A}^c$. That is, $x_n \notin A$. This is a contradiction. Thus, we must have that $x \in \overline{A}$. □

The reason the above theorem is so consequential, in large part, is because it gives us the next two theorems.

Theorem 5.1.6. *Let (X, τ) be a first countable topological space.*

(i) $O \in \tau$ if and only if, for every $x \in O$, if $(x_n)_{n=1}^{\infty}$ is a sequence in X such that $x_n \xrightarrow{\tau} x$, then there exists $N \in \mathbb{Z}_+$ such that, for all $n \geq N$, we have $x_n \in O$.

(ii) C is closed in X with respect to τ if and only if, for every sequence $(x_n)_{n=1}^{\infty}$ in C, if $x_n \xrightarrow{\tau} x$, then $x \in C$.

Proof. We will start by proving statement (i). The forward direction follows immediately from the definition of a convergent sequence. For the backward direction, let O be a subset of X and suppose that, for every $x \in O$, if $(x_n)_{n=1}^{\infty}$ is a sequence in X such that $x_n \to x$, then there exists $N \in \mathbb{Z}_+$ such that, for all $n \geq N$, we have $x_n \in O$.

Now, suppose O is not open. Then, by Exercise **??**, we have that $O^o \subset O$. Let $x \in O \setminus O^o$. Note then there does not exist an open set U such that $x \in U \subseteq O$, otherwise $x \in O^o$. Let $\mathcal{B}_x = \{B_1, B_2, \dots\}$ be a countable neighborhood base for x. Without loss of generality, we can assume $B_k \supseteq B_{k+1}$ for all $k \in \mathbb{Z}_+$ by following the same procedure that was used in the proof of Theorem 5.1.5.

Claim: For all $n \in \mathbb{Z}_+$, we have that $B_n \cap O^c \neq \emptyset$. If $B_n \cap O^c = \emptyset$ for some $n \in Z_+$, then $B_n \subseteq O$ and B_n is an open neighborhood of x. Thus, $x \in B_n \subseteq O^o$, contradicting the fact that $x \notin O^o$. So, we must have that $B_n \cap O^c \neq \emptyset$, for all $n \in Z_+$.

Let $x_n \in B_n \cap O^c$ for all $n \in \mathbb{Z}_+$. Then $x_n \to x$. By our assumption, there exists $N \in Z_+$ such that, for all $n \geq N$, we have $x_n \in O$. But, $x_n \in O^c$ for all $n \in Z_+$ so we arrive at a contradiction. Thus, we must have that $O = O^o$ and so $O \in \tau$.

To prove the forward direction of statement (ii), suppose C is a closed set and let $(x_n)_{n=1}^\infty$ be a sequence in C such that $x_n \to x$, for some $x \in X$. Then, we have that x is a limit point for C and, since C is closed, by Theorem **??**, we have that $x \in C$.

For the other direction, suppose that for every sequence $(x_n)_{n=1}^\infty$ in C, if $x_n \to x$, then $x \in C$. We want to show $C = \overline{C}$. Since we always have that $C \subseteq \overline{C}$, it suffices to prove the reverse inclusion. Let $x \in \overline{C}$. Then, by Theorem **??**, either $x \in C$ and we are done or x is a limit point of C, so suppose x is a limit point of C. Since (X, τ) is first countable, there is a countable open neighborhood base $B_x = \{B_1, B_2, \dots\}$ for x. Without loss of generality, suppose $B_k \supseteq B_{k+1}$ for all $k \in Z_+$. Since x is a limit point of C, we have that $B_k \cap C \neq \emptyset$ for all $k \in Z_+$ so pick $x_n \in B_n \cap C$, for all $n \in \mathbb{Z}_+$. Then $(x_n)_{n=1}^\infty$ is a sequence in C and $x_n \to x$. By our assumption, we have that $x \in C$. Hence, $C = \overline{C}$ and so, by Theorem **??**, we have that C is closed. □

Exercise 5.1.7. Let X be a set and let τ_1 and τ_2 be topologies on X and suppose (X, τ_1) is first countable. Prove that $\tau_2 \subseteq \tau_1$ if and only if every sequence in X which converges with respect to τ_1 to some $x \in X$ also converges to x with respect to τ_2.

Theorem 5.1.8. *Let (X, τ) and (Y, σ) be first countable topological spaces and let $f : (X, \tau) \to (Y, \sigma)$. Then f is continuous if and only if whenever $(x_n)_{n=1}^\infty$ is a* *sequence in X such that $x_n \xrightarrow{\tau} x$, for some $x \in X$, we have that $f(x_n) \xrightarrow{\sigma} f(x)$.*

Proof. For the forward direction, suppose f is continuous. Let $(x_n)_{n=1}^{\infty}$ be a sequence in X such that $x_n \to x$, for some $x \in X$. Let O be an open neighborhood of $f(x)$. Since f is continuous and $f(x) \in O$, we have that $f^{-1}(O)$ is an open neighborhood of x. Since $x_n \to x$, there exists $N \in Z_+$ such that, for all $n \geq N$, we have $x_n \in f^{-1}(O)$. Then, for all $n \geq N$, we have $f(x_n) \in O$ and so $f(x_n) \to f(x)$.

For the other direction, suppose f is not continuous. Then, for some $x_0 \in X$, the function f is not continuous at x_0. Thus, there exists an open neighborhood O of $f(x_0)$ such that, for all open neighborhoods U of x_0, we have that $f(U) \not\subseteq O$. Let $\mathcal{B}_{x_0} = \{B_1, B_2, \ldots\}$ be a countable neighborhood base for x_0 and, without loss of generality, suppose $B_k \supseteq B_{k+1}$ for all $k \in Z_+$. Since B_k is an open neighborhood of x_0, we have that $f(B_k) \not\subseteq O$. Hence, there exists $x_k \in B_k$ such that $f(x_k) \notin O$, for all $k \in \mathbb{Z}_+$. Then we have that $x_n \to x_0$ but $f(x_n)$ does not converge to $f(x_0)$. □

The next exercise will be used in the following example.

Exercise 5.1.9. Let (X, τ) be a topological space and let $\mathcal{B}$ be a base for τ. Let $A \subseteq X$ and let $x \in X$. Suppose that for every $B \in \mathcal{B}$ such that $x \in B$, we have that $B \cap A \neq \emptyset$. Prove that $x \in \overline{A}$. *Hint: Suppose $x \notin \overline{A}$. Then there exists a closed set C such that $A \subseteq C$ and $x \notin C$. Why? Now show there exists a basic open set B such that $x \in B \subseteq X \setminus A$. Explain why this gives us a contradiction.*

Example 5.1.10. Consider $X = \mathbb{R}^{\mathbb{R}}$ with the product topology τ. Let A be the set of all $f \in X$ such that

$$f(x) = \begin{cases} 0 & \text{if } x \in F_f \\ 1 & \text{otherwise} \end{cases}$$

for some finite set F_f. If $g : \mathbb{R} \to \mathbb{R}$ where $g(x) = 0$ for all $x \in \mathbb{R}$, then $g \notin A$ but $g \in \overline{A}$. Indeed, let O be a basic open neighborhood of g. Then there exists $n \in \mathbb{Z}_+$,

$x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$, and $\epsilon_1, \ldots, \epsilon_n \in (0, \infty)$ such that

$$O = \pi_{x_1}^{-1}(B(y_1, \epsilon_1)) \cap \cdots \cap \pi_{x_n}^{-1}(B(y_n, \epsilon_n)).$$

Let $F_f = \{x_1, \ldots, x_n\}$ and define $f(x) = 0$ if $x \in F_f$ and $f(x) = 1$ if $x \notin F_f$. Then $f \in A$ and $f(x_k) = g(x_k)$ for all $k = 1, \ldots, n$. Then, since $g \in O$ we have that $f \in O$. Hence, for any basic open neighborhood O of g, we can find $f \in A \cap O$. Thus, by the exercise above, $g \in \overline{A}$.

However, there exists no sequence $(f_n)_{n=1}^{\infty}$ in A such that $f_n \to g$. Suppose so. Let $F = \cup_{n=1}^{\infty} F_{f_n}$. Then F is a countable union of finite sets so F is a countable subset of $\mathbb{R}$. Hence, $\mathbb{R} \setminus F \neq \emptyset$. Let $x \in \mathbb{R} \setminus F$. Note then that $f_n(x) = 1$ for all $n \in \mathbb{Z}_+$. Then $B = \pi_x^{-1}(B(0, \frac{1}{2})$ is a basic open neighborhood of g but, for all $n \in \mathbb{Z}_+$, the function $f_n \notin B$ since $|f_n(x) - 0| = 1 > \frac{1}{2}$. Therefore, the sequence $(f_n)_{n=1}^{\infty}$ cannot converge to g. Hence, we have shown that while $g \in \overline{A}$, there exists no sequence in A which converges to it.

This example illustrates several things. First, it shows that Theorem 5.1.5 is not true for a general topological space. It also shows, again from Theorem 5.1.5, that $\mathbb{R}^{\mathbb{R}}$ with the product topology is not first countable. Further, this proves a statement which was made in Chapter 2. Recall it was stated in Chapter 2 that there exists no metric d on $\mathbb{R}^{\mathbb{R}}$ such that a sequence in $\mathbb{R}^{\mathbb{R}}$ would converge pointwise if and only if it converged with respect to d. We now know this to be the case. For if there were such a metric, it would have to generate the product topology on $\mathbb{R}^{\mathbb{R}}$. However, this would mean the product topology on $\mathbb{R}^{\mathbb{R}}$ is first countable, as all topologies induced by metrics are first countable. Since $\mathbb{R}^{\mathbb{R}}$ is not first countable, there cannot exist such a metric.

When a topology τ on a set X cannot be induced by a metric, then we say the topological space (X, τ) is **nonmetrizable**. We now see that $\mathbb{R}^{\mathbb{R}}$ with the product topology is nonmetrizable. The above argument also shows that any topological space which is not first countable is nonmetrizable. For example, we mentioned earlier that $\mathbb{R}$ with the finite complement topology is not first countable, so it too is nonmetrizable. This is proof then that taking the step in abstraction from metric spaces to general topological spaces was a necessary one.

5.2 Nets

As we have now seen, there are topological spaces where sequences are not a sufficient tool for studying their topological structure. Can we find a way to generalize sequences to other mathematical objects more suited for these types of topological spaces? Recall that one way to view a sequence $x = (x_n)_{n=1}^{\infty}$ in a set X is to view $x : \mathbb{Z}_+ \to X$, where $x(k) = x_k$. One way to generalize a sequence is to then lessen the requirement that the domain of x be $\mathbb{Z}_+$. Of course, letting the domain be any set takes the generalization a bit too far. We would like to preserve certain properties of $\mathbb{Z}_+$ for our new domain. Specifically, we would like to preserve certain properties of the usual ordering we have on $\mathbb{Z}_+$, namely, reflexivity and transitivity. There is also a third property of this ordering on $\mathbb{Z}_+$ that is fundamental to the concept and usefulness of sequences: given $k, n \in \mathbb{Z}_+$, we can always find $m \in \mathbb{Z}_+$ such that $k \leq m$ and $n \leq m$. Replacing the set $\mathbb{Z}_+$ with a set which has a relation satisfying these properties is how we arrive at the definition of nets.

Definition 5.2.1. A set Λ is a **directed set** if there is a relation $\leq$ on Λ such that

(i) $\leq$ is reflexive,

(ii) $\leq$ is transitive, and

(iii) for all $\lambda_1, \lambda_2 \in \Lambda$, there exists $\lambda_3 \in \Lambda$ such that $\lambda_1 \leq \lambda_3$ and $\lambda_2 \leq \lambda_3$.

Example 5.2.2. (i) Obviously, the set $\mathbb{Z}_+$ is a directed set with its usual ordering. Recall that it is our motivation for defining directed sets in the first place.

(ii) The set $\mathbb{R}$ with the usual ordering is a directed set.

(iii) Let X be a nonempty set and, for any $A, B \in \mathcal{P}(X)$, let $A \leq B \Leftrightarrow A \subseteq B$. Then $\mathcal{P}(X)$ is a directed set with this ordering. The proof is a following exercise.

Exercise 5.2.3. Let X be a nonempty set and define a relation $\leq$ on $\mathcal{P}(X)$ by $A \leq B \Leftrightarrow A \subseteq B$. Prove $\mathcal{P}(X)$ with the relation $\leq$ is a directed set.

Definition 5.2.4. Let X be a nonempty set and let Λ be a directed set. A **net** in the set X is a function $x : \Lambda \to X$. Typically, we write $x(\lambda) = x_\lambda$ and refer to x as the net $(x_\lambda)_{\lambda \in \Lambda}$.

Example 5.2.5. (i) If X is a nonempty set, then every sequence in X is also a net in X. A sequence is a specific kind of net where we take the directed set Λ to be $\mathbb{Z}_+$ with the usual ordering.

(ii) Let $\Lambda = \mathcal{P}(\mathbb{Z}_+) \setminus \{\emptyset\}$ and define a relation on Λ by $A \leq B$ if and only if $A \subseteq B$. Then Λ with this relation is a directed set. Now, for each $A \in \Lambda$, define $x_A = \min A$. By the well-ordering principle, x_A is well-defined. Then $(x_A)_{A \in \Lambda}$ is a net in $\mathbb{Z}_+$.

Definition 5.2.6. Let (X, τ) be a topological space and let $(x_\lambda)_{\lambda \in \Lambda}$ be a net in X. We say the net $(x_\lambda)_{\lambda \in \Lambda}$ **converges** to some $x \in X$ if, for each open neighborhood U of x there exists $\lambda_0 \in \Lambda$ such that, if $\lambda \geq \lambda_0$, then $x_\lambda \in U$. In this case we write $x_\lambda \xrightarrow{\lambda} x$.

Note that in the case when $\lambda = \mathbb{Z}_+$, the above definition is precisely the definition for a convergent sequence.

Example 5.2.7. Let (X, τ) be a topological space and let $x_0 \in X$. Let Λ be the collection of all elements of τ which contain x_0. Define a relation $\leq$ on Λ by $U_1 \leq U_2$ if and only if $U_1 \supseteq U_2$. Then Λ with this relation is a directed set. Now, since each element of Λ is nonempty, for each $U \in \Lambda$, pick $x_U \in U$. Then $(x_U)_{U \in \Lambda}$ is a net in X. Further, $x_U \xrightarrow{U} x_0$.

Just as subsequences play an important role in analysis, subnets will also play an important role in general topological spaces. Let's first recall the notion of a subsequence.

Definition 5.2.8. Let X be a set and let $(x_n)_{n=1}^{\infty}$ be a sequence in X. Let $n : \mathbb{Z}_+ \to \mathbb{Z}_+$ be a strictly increasing function, where we denote $n(k) = n_k$, and define $y_k = x_{n_k}$. Then we say the sequence $(y_k)_{k=1}^{\infty}$ is a **subsequence** of $(x_n)_{n=1}^{\infty}$. Note that we usually do not relabel the subsequence and instead reference the subsequence $(x_{n_k})_{n=1}^{\infty}$ instead of $(y_k)_{k=1}^{\infty}$.

Example 5.2.9. Let $(x_n)_{n=1}^{\infty} = (1, 2, 3, 4, \dots)$. Then a subsequence of $(x_n)_{n=1}^{\infty}$ would be, for example, $(x_{n_k})_{n=1}^{\infty} = (2, 4, 6, 8, \dots)$. Formally, the map $n : \mathbb{Z}_+ \to \mathbb{Z}_+$ would be $n_k = n(k) = 2k$. With this notation then, $x_{n_1} = 2$, $x_{n_2} = 4$, etc.

Definition 5.2.10. (i) A subset Λ' of a directed set Λ is called **cofinal** if for every $\lambda \in \Lambda$ there exists $\lambda' \in \Lambda'$ such that $\lambda \leq \lambda'$.

(ii) If $(M, \leq)$ and $(\Lambda, \leq)$ are two directed sets, then a function $h : M \to \Lambda$ is called **increasing** if, for all $\mu_1, \mu_2 \in M$, if $\mu_1 \leq \mu_2$, then $h(\mu_1) \leq h(\mu_2)$.

(iii) If $(x_\lambda)_{\lambda\in\Lambda}$ is a net in a set X, then a net $(y_\mu)_{\mu\in M}$ is called a **subnet** of $(x_\lambda)_{\lambda\in\Lambda}$ if there exists an increasing function λ from the directed set $(M, \leq)$ to the directed set $(\Lambda, \leq)$ such that $\lambda(M)$ is a cofinal subset of Λ, and $y_\mu = x_{\lambda(\mu)}$. Note that we typically write λ_μ instead of $\lambda(\mu)$. As with subsequences, we typically do not relabel the subnet so we usually reference the subnet $(x_{\lambda_\mu})_{\mu\in M}$ rather than $(y_\mu)_{\mu\in M}$.

Example 5.2.11. Let (X, τ) be a topological space with base $\mathcal{B}$ and let $x_0 \in X$. As in Example 5.2.7, let Λ be the collection of all elements of τ which contain x_0. Define a relation on $\leq$ on Λ by $\lambda_1 \leq \lambda_2$ if and only if $\lambda_1 \supseteq \lambda_2$. Then Λ with this relation is a directed set. Now, since each element of Λ is nonempty, for each $\lambda \in \Lambda$, pick $x_\lambda \in \lambda$. Then $(x_\lambda)_{\lambda\in\Lambda}$ is a net in X. If we define M to be the collection of all elements of Λ which come from β then M is also a directed set with the same relation as Λ. Note that the map $\lambda : M \to \Lambda$ defined by $\lambda(\mu) = \mu$ is increasing. The set $\lambda(M) = M$ is also cofinal in Λ since, for any $\lambda_0 \in \Lambda$, since $\mathcal{B}$ is a base for τ, there exists $\mu \in M$ such that $\mu \subseteq \lambda_0$. That is, $\mu \geq \lambda_0$. Hence, $(x_{\lambda_\mu})_{\mu \in M}$ is a subnet of $(x_\lambda)_{\lambda \in \Lambda}$. Note also that $x_\lambda \xrightarrow{\lambda} x_0$ and $x_{\lambda_\mu} \xrightarrow{\mu} x_0$.

Similar to how every subsequence of a convergent sequence is convergent, every subnet of a convergent net converges, as the next proposition shows.

Proposition 5.2.12. *If $(x_\lambda)_{\lambda \in \Lambda}$ is a net in a topological space (X, τ) such that $x_\lambda \xrightarrow{\lambda} x$, for some $x \in X$. Then every subnet $(x_{\lambda_\mu})_{\mu \in M}$ of $(x_\lambda)_{\lambda \in \Lambda}$ converges to x.*

Proof. Let $(x_\lambda)_{\lambda \in \Lambda}$ be a net in X such that $x_\lambda \xrightarrow{\lambda} x$, for some $x \in X$. Let $(x_{\lambda_\mu})_{\mu \in M}$ be a subnet of $(x_{\lambda_\mu})_{\mu \in M}$ and let O be an open neighborhood of x. Since $x_\lambda \xrightarrow{x}$, there exists $\lambda_0 \in \Lambda$ such that, for all $\lambda \geq \lambda_0$, we have that $x_\lambda \in O$. Since $\lambda(M)$ is cofinal in Λ, there exists $\mu_0 \in M$ such that $\lambda_{\mu_0} \geq \lambda_0$. Further, since $\lambda : M \to \Lambda$ is increasing, for all $\mu \geq \mu_0$ we have that $\lambda_\mu \geq \lambda_{\mu_0} \geq \lambda_0$ and so $x_{\lambda_\mu} \in O$. $\square$

If you have studied analysis, then you have probably noticed that subsequences show up, most often, in an abstract setting. We don't often have to deal with specific subsequences. This is even more true for subnets. We almost exclusively find ourselves dealing with generic subnets rather than specific ones.

Definition 5.2.13. Let (X, τ) be a topological space. Let $(x_\lambda)_{\lambda \in \Lambda}$ be a net in X and let $x \in X$. We say x is a **cluster point** of $(x_\lambda)_{\lambda \in \Lambda}$ if, for each open neighborhood U of x and for each $\lambda_0 \in \Lambda$, there exists $\lambda \in \Lambda$ such that $\lambda \geq \lambda_0$ and $x_\lambda \in U$.

Example 5.2.14. If $(x_n)_{n=1}^{\infty}$ is the sequence in $\mathbb{R}$ defined by $x_n = (-1)^n$, then 1 and -1 are cluster points for the sequence.

Theorem 5.2.15. *Let (X, τ) be a topological space and let $(x_\lambda)_{\lambda \in \Lambda}$ be a net in X. A point $x \in X$ is a cluster point for $(x_\lambda)_{\lambda \in \Lambda}$ if and only if there exists a subnet $(x_{\lambda_\mu})_{\mu \in M}$ of $(x_\lambda)_{\lambda \in \Lambda}$ such that $x_{\lambda_\mu} \xrightarrow{\mu} x$.*

Proof. Let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in X.

For the forward direction, suppose x is a cluster point for $(x_\lambda)_{\lambda\in\Lambda}$. Let

$$M = \{(\lambda, U) |\, \lambda \in \Lambda\,,\, U \text{ is an open nbd of } x \text{ such that } x_\lambda \in U\}.$$

Define a relation $\leq$ on M by $(\lambda_1, U_1) \leq (\lambda_2, U_2)$ if and only if $\lambda_1 \leq \lambda_2$ and $U_2 \subseteq U_1$.

Claim: The pair $(M, \leq)$ is a directed set.

Obviously, $(M, \leq)$ is reflexive and transitive. Let $(\lambda_1, U_1), (\lambda_2, U_2) \in M$. Since Λ is a directed set, there exists $\lambda_0 \in \Lambda$ such that $\lambda_1 \leq \lambda_0$ and $\lambda_2 \leq \lambda_0$. Since U_1 and U_2 are open neighborhoods of x, we have that $U_3 = U_1 \cap U_2$ is an open neighborhood of x. Since x is a cluster point, there exists $\lambda_3 \geq \lambda_0$ such that $x_{\lambda_3} \in U_3$. Hence, $(\lambda_3, U_3) \in M$ and $(\lambda_1, U_1) \leq (\lambda_3, U_3)$ and $(\lambda_2, U_2) \leq (\lambda_3, U_3)$.

Define $h : M \to \Lambda$ by $h(\lambda, U) = \lambda$. The function h is obviously increasing and cofinal. Let U_0 be an open neighborhood of x. Since x is a cluster point, pick $\lambda_0 \in \Lambda$ such that $x_{\lambda_0} \in U_0$. Then $(\lambda_0, U_0) \in M$ and, if $(\lambda, U) \geq (\lambda_0, U_0)$, then $x_{(\lambda,U)} = x_\lambda \in U \subseteq U_0$. Hence, $x_{(\lambda,U)} \to x$.

For the other direction, suppose $(x_{\lambda_\mu})_{\mu\in M}$ is a subnet of $(x_\lambda)_{\lambda\in\Lambda}$ and $x_{\lambda_\mu} \to x$. Let U be an open neighborhood of x. Let $\lambda_0 \in \Lambda$. Since $x_{\lambda_\mu} \to x$, there exists $\mu_0 \in M$ such that, for all $\mu \geq \mu_0$, we have $x_{\lambda_\mu} \in U$. Pick $\lambda_1 \in \Lambda$ such that $\lambda_1 \geq \lambda_0$ and $\lambda_1 \geq \lambda_{\mu_0}$. Then pick $\lambda_{\mu_1} \geq \lambda_1$ and we have that $x_{\lambda_{\mu_1}} \in U$. Hence, x is a cluster point for $(x_\lambda)_{\lambda\in\Lambda}$. □

Corollary 5.2.16. *Let (X, τ) be a topological space. Let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in X and let $(x_{\lambda_\mu})_{\mu\in M}$ be a subnet of $(x_\lambda)_{\lambda\in\Lambda}$. If x is a cluster point of $(x_{\lambda_\mu})_{\mu\in M}$, then x is a cluster point of $(x_\lambda)_{\lambda\in\Lambda}$.*

Proof. If x is a cluster point of $(x_{\lambda_\mu})_{\mu\in M}$, by Theorem 5.2.15, there exists a subnet of $(x_{\lambda_\mu})_{\mu\in M}$ which converges to x. But a subnet of $(x_{\lambda_\mu})_{\mu\in M}$ is also a subnet of $(x_\lambda)_{\lambda\in\Lambda}$

so again, by Theorem 5.2.15, we have that x is a cluster point of $(x_\lambda)_{\lambda\in\Lambda}$. □

The next theorem shows that we get a version Theorem 5.1.5 for a general topological space if we replace sequences with nets.

Theorem 5.2.17. *Let (X,τ) be a topological space and let $A \subseteq X$. Then a point $x \in \overline{A}$ if and only if there exists a net $(x_\lambda)_{\lambda\in\Lambda}$ in A such that $x_\lambda \xrightarrow{\tau} x$.*

Proof. For the forward direction, let $x \in \overline{A}$. Then, for all open neighborhoods U of x, we have that $U \cap A \neq \emptyset$. Let λ be the set of all open neighborhoods of x where we define the relation $\leq$ by $U_1 \leq U_2$ if and only if $U_1 \supseteq U_2$. Then, for all $U \in \Lambda$, pick $x_U \in U \cap A$. Then $(x_U)_{U\in\Lambda}$ is a net in A and $x_U \to x$.

For the other direction, let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in A such that $x_\lambda \to x$. Let O be an open neighborhood of x. Then there exists $\lambda_0 \in \Lambda$ such that, for all $\lambda \geq \lambda_0$, we have that $x_\lambda \in O$. Thus, $x_\lambda \in O \cap A$ and so $O \cap A \neq \emptyset$. Hence, x is a limit point of A and so $x \in \overline{A}$. □

The next theorem shows that we get a version of Theorem 5.1.6 for a general topological space if we replace sequences with nets.

Theorem 5.2.18. *Let (X,τ) be a first countable topological space.*

(i) $O \in \tau$ if and only if, for every $x \in O$, if $(x_\lambda)_{\lambda\in\Lambda}$ is a net in X such that $x_\lambda \xrightarrow{\tau} x$, then there exists $\lambda_0 \in \Lambda$ such that, for all $\lambda \geq \lambda_0$, we have $x_\lambda \in O$.

(ii) C is closed in X with respect to τ if and only if, for every net $(x_\lambda)_{\lambda\in\Lambda}$ in C, if $x_\lambda \xrightarrow{\tau} x$, then $x \in C$.

Proof. need to figure this one out still-can remove it since it isn't used later as long as the next exercise is removed □

A useful consequence of the above theorem is given in the next exercise.

Exercise 5.2.19. Let X be a set and let τ_1 and τ_2 be topologies on X. Prove $\tau_2 \subseteq \tau_1$ if and only if every net $(x_\lambda)_{\lambda\in\Lambda}$ in X which converges with respect to τ_1 to some $x \in X$ also converges to x with respect to τ_2.

We also obtain a result similar to Theorem 5.1.8 for general topological spaces if we replace sequences with nets.

Theorem 5.2.20. *Let (X, τ) and (Y, σ) be topological spaces and let $f : (X, \tau) \to (Y, \sigma)$. Then f is continuous at x_0 if and only if, whenever $x_\lambda \xrightarrow{\tau} x_0$ in X, we have that $f(x_\lambda) \xrightarrow{\sigma} f(x_0)$ in Y.*

Proof. Suppose f is continuous at x_0. Let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in X such that $x_\lambda \to x_0$. Let U be an open neighborhood of $f(x_0)$. Then $f^{-1}(U)$ is an open neighborhood of x_0. Since $x_\lambda \to x_0$, there exists $\lambda_0 \in \Lambda$ such that, for all $\lambda \geq \lambda_0$, we have that $x_\lambda \in f^{-1}(U)$. Then, for all $\lambda \geq \lambda_0$, we have that $f(x_\lambda) \in U$ and so $f(x_\lambda) \to f(x_0)$.

For the other direction, suppose f is not continuous at x_0. Then, for some open neighborhood V of $f(x_0)$, we have that $f(U) \not\subseteq V$, for all open neighborhoods U of x_0. Let Λ be the set of all open neighborhoods of x_0 and define $\leq$ on Λ by $U_1 \leq U_2$ if and only if $U_1 \supseteq U_2$. Since $f(U) \not\subseteq V$ for all $U \in \Lambda$, let $x_U \in U$ such that $f(x_U) \notin V$, for all $U \in \Lambda$. Then $(x_U)_{U\in\Lambda}$ is a net in X and $x_U \to x_0$ but $f(x_U)$ does not converge to $f(x_0)$ since V is an open neighborhood of x_0 but $f(x_U) \notin V$, for all $U \in \Lambda$. □

An important consequence of the above theorem is the following theorem.

Theorem 5.2.21. *Let X be a set and let (X_i, τ_i) be topological spaces for all $i \in I$. For each $i \in I$, let $f_i : X \to X_i$ and set $\mathcal{F} = \{f_i | \, i \in I\}$. Let τ be the weak topology generated on X by $\mathcal{F}$. A net $(x_\lambda)_{\lambda\in\Lambda}$ converges to $x \in X$ if and only if $f_i(x_\lambda) \xrightarrow{\lambda} f_i(x)$ for all $i \in I$.*

Proof. Let $i \in I$ and suppose $x_\lambda \xrightarrow{\lambda} x$. Since τ is the weakest topology on X which makes all the elements of $\mathcal{F}$ continuous, $f_i : (X, \tau) \to (X_i, \tau_i)$ is continuous. Hence, by Theorem 5.2.20, we have that $f_i(x_\lambda) \xrightarrow{\tau} f_i(x)$.

For the other direction, suppose $f_i(x_\lambda) \xrightarrow{\tau} f_i(x)$ for all $i \in I$. Let B be a basic open neighborhood of x in X. Then there exists $n \in \mathbb{Z}_+$, $i_1, \ldots, i_n \in I$ and $O_{i_k} \in \tau_{i_k}$, for all $k = 1, \ldots, n$, such that

$$B = f_{i_1}^{-1}(O_{i_1}) \cap f_{i_2}^{-1}(O_{i_2}) \cap \cdots \cap f_{i_n}^{-1}(O_{i_n}).$$

Since B is an open neighborhood of x, we have that $x \in f_{i_k}^{-1}(O_{i_k})$ for all $k = 1, \ldots, n$, and so $f_{i_k}(x) \in O_{i_k}$, for all $k = 1, \ldots n$. By our assumption, for each $k = 1, \ldots, n$, there exists λ_k such that, if $\lambda \geq \lambda_k$, then $f_{i_k}(x_\lambda) \in O_{i_k}$. Pick λ_0 to be greater than or equal to all $\lambda_1, \ldots, \lambda_n$. Then, for $\lambda \geq \lambda_0$, we have that $f_{i_k}(x_\lambda) \in O_{i_k}$ for all $i = 1, \ldots n$ and so $x_\lambda \in B$. □

Recall that the product topology on a product space can be viewed as the weak topology generated by the projection maps. Hence, the above theorem immediately implies the following corollary.

Corollary 5.2.22. *Let (X_i, τ_i) be a topological space for all $i \in I$. Let $X = \prod_{i \in I} X_i$ with the product topology τ and let $(x_\lambda)_{\lambda \in \Lambda}$ be a net in X. Then $(x_\lambda)_{\lambda \in \Lambda}$ converges to $x \in X$ if and only if $\pi_i(x_\lambda) \xrightarrow{\tau} \pi_i(x)$ for all $i \in I$.*

As we have already discussed, often times, the collection of topological spaces (X_i, τ_i) for all $i \in I$, are all the same, say (Y, σ). The product space $X = \prod_{i \in I} X_i$ then becomes Y^I, or rather, the set of functions $f : I \to Y$. In this setting, the above corollary tells us that a net $(x_\lambda)_{\lambda \in \Lambda}$ in Y^I converges to $x \in Y^I$ if and only if $x_\lambda(i) \xrightarrow{\sigma} x(i)$ for all $i \in I$ since $\pi_i(x_\lambda) = x_\lambda(i)$ for all $i \in I$.

To be even more precise, we have that if $\mathbb{R}^\mathbb{R}$ is equipped with the product topology, then a net $(f_\lambda)_{\lambda \in \Lambda}$ converges to some $f \in \mathbb{R}^\mathbb{R}$ if and only if $f_\lambda(x) \to f(x)$ for all

$x \in \mathbb{R}$. Hence, nets in $\mathbb{R}^{\mathbb{R}}$ converge precisely when we have pointwise convergence. Furthermore, since we have already seen that $\mathbb{R}^{\mathbb{R}}$ is not first countable, to use theorems 5.2.17 and 5.2.20 for $\mathbb{R}^{\mathbb{R}}$, we have to investigate the pointwise convergence of nets. For example, suppose we have a set A in $\mathbb{R}^{\mathbb{R}}$ and we want to show that $f \in \overline{A}$. It suffices to show that there exists a net $(f_\lambda)_{\lambda\in\Lambda}$ in A such that $f_\lambda(x) \to f(x)$ for all $x \in \mathbb{R}$. The next example illustrates a standard method for proving a function $T : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}$ is continuous when $\mathbb{R}^{\mathbb{R}}$ has the product topology.

Example 5.2.23. Fix $t \in \mathbb{R}$ and consider the function $T : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}$ defined by $T(f)(x) = f(x+t)$ for all $x \in \mathbb{R}$. Assume $\mathbb{R}^{\mathbb{R}}$ has the product topology in the domain and codomain. To prove that T is continuous, by Theorem 5.2.20 and the discussion above, it is enough to show that if $(f_\lambda)_{\lambda\in\lambda}$ is a net in $\mathbb{R}^{\mathbb{R}}$ and there exists $f \in \mathbb{R}^{\mathbb{R}}$ such that $f_\lambda(x) \to f(x)$ for all $x \in \mathbb{R}$, then we have that $T(f_\lambda)(x) \to T(f)(x)$ for all $x \in \mathbb{R}$. In this case, checking that $T(f_\lambda)(x) \to T(f)(x)$ is fairly trivial since it is equivalent to checking that $f_\lambda(x+t) \to f(x+t)$ which we know is the case since $t + x \in \mathbb{R}$ and we are assuming that $f_\lambda(y) \to f(y)$ for all $y \in \mathbb{R}$ (just take $y = x + t$). Thus, the function T is continuous.

The next two exercises are both standard homework problems in an analysis class for sequences in $\mathbb{R}$ (or $\mathbb{R}^n$). Here, we are able to prove them in more generality for nets in general topological spaces.

Exercise 5.2.24. Let (X, τ) be a topological space and suppose $(x_\lambda)_{\lambda\in\Lambda}$ is a net in X such that $x_\lambda \xrightarrow{\lambda} x$, for some $x \in X$. Prove every subnet of $(x_\lambda)_{\lambda\in\Lambda}$ also converges to x.

Exercise 5.2.25. Let (X, τ) be a topological space and suppose $(x_\lambda)_{\lambda\in\Lambda}$ is a net in X. Let $x \in X$. Prove that if every subnet of $(x_\lambda)_{\lambda\in\Lambda}$ has a subnet which converges to x, then $x_\lambda \xrightarrow{\lambda} x$. *Hint: Suppose not. That is, suppose there exists an open neighborhood O of x such that, for all $\lambda_0 \in \Lambda$, there exists $\lambda \geq \lambda_0$ such that $x_\lambda \notin O$. Define $M = \{\lambda \in \Lambda \,|\, x_\lambda \notin O\}$. For $\mu \in M$, define $\lambda_\mu = \mu$. Thus, $(x_{\lambda_\mu})_{\mu\in M}$ is a subnet of $(x_\lambda)_{\lambda\in\Lambda}$. Now find a contradiction.*

Index