MATH 422-Introduction to Topology

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4.1 The Subspace Topology

Suppose (X, d) is a metric space and let $Y \subseteq X$. There is a very obvious way to then define a metric d_0 on Y. Simply, let $d_0 = d|_{Y \times Y}$, since $Y \times Y \subseteq X \times X$. Then (Y, d_0) is a metric space and the distance we have between two elements of Y agrees with the distance between the same two elements when considered elements of X. Now, what is the appropriate way to generalize this idea for general topological spaces? To be precise, suppose (X, τ) is a topological space and $Y \subseteq X$. What is the natural way to then define a topology τ_0 on Y which "agrees" with τ?

Let's reexamine the situation above with the metric spaces. Let O be an open set in (Y, d_0). Since the open balls form a base for the topology induced on Y by the metric d_0, there exists an index set I, $\{y_i | i \in I\} \subseteq Y$, and $\{r_i | i \in I\} \subseteq (0, \infty)$ such that $O = \bigcup_{i \in I} B_Y(y_i, r_i)$, where $B_Y(y_i, r_i) = \{y \in Y | d_0(y_i, y) < r_i\}$. If we denote $B_X(y_i, r_i) = \{x \in X | d(y_i, x) < r_i\}$ then, for all $i \in I$, we have that $B_Y(y_i, r_i) = B_X(y_i, r_i) \cap Y$. Hence,

$$O = \bigcup_{i \in I} B_Y(y_i, r_i) = \bigcup_{i \in I} (B_X(y_i, r_i) \cap Y) = \left(\bigcup_{i \in I} B_X(y_i, r_i) \right) \cap Y$$

and $\bigcup_{i \in I} B_X(y_i, r_i)$ is an open set in (X, d). Hence, any open set O in (Y, d_0) is of the form $O = U \cap Y$ for some open set U in (X, d). Further, sets of the form $U \cap Y$, where U is open in (X, d), form a topology on Y. This is then precisely how we want a subset of a topological space to inherit a topology from its superset! The details are given in the next proposition and definition.

Proposition 4.1.1. *Let (X, τ) be a topological space and let $Y \subseteq X$. Define $\tau|_Y = \{O \cap Y | O \in \tau\}$. Then $\tau|_Y$ is a topology on Y.*

Proof. Since $\emptyset, X \in \tau$, we have that $\emptyset = \emptyset \cap Y \in \tau|_Y$ and $Y = X \cap Y \in \tau|_Y$. Further, let $O_i \cap Y \in \tau|_Y$ for all $i \in I$, where $O_i \in \tau$. Then $\cup_{i \in I}(O_i \cap Y) = (\cup_{i \in I} O_i) \cap Y \in \tau|_Y$, since $\cup_{i\in I} O_i \in \tau$. Lastly, suppose $O_1 \cap Y, O_2 \cap Y, \ldots, O_n \cap Y \in \tau|_Y$ for some $n \in \mathbb{Z}_+$ and $O_1, O_2, \ldots, O_n \in \tau$. Then $\cap_{i=1}^n (O_i \cap Y) = (\cap_{i=1}^n O_i) \cap Y \in \tau|_Y$ since $\cap_{i=1}^n O_i \in \tau$. Therefore, $\tau|_Y$ is a topology on Y. □

Definition 4.1.2. Let (X, τ) be a topological space and let $Y \subseteq X$. The **relative topology** or **subspace topology** on Y is given by $\tau|_Y = \{O \cap Y | O \in \tau\}$.

Remark 4.1.3. Let (X, τ) be a topological space and let $Y \subseteq X$. If we refer to Y as a **subspace** of X then it is understood that Y has the relative topology. Often times in the literature, authors will discuss topological properties of subsets of topological spaces without specifying a topology because they are understood to have the relative topology.

Example 4.1.4. (i) Consider $X = \mathbb{R}$ with the usual topology τ and $Y = [0, 10]$. Examples of open sets in Y with the relative topology would then be

(a) $(1, 2)$ since $(1, 2) = (1, 2) \cap [0, 10]$ and $(1, 2) \in \tau$,

(b) $(5, 10]$ since $(5, 10] = (5, 15) \cap [0, 10]$ and $(5, 15) \in \tau$, and

(c) $[0, 1)$ since $[0, 1) = (-1, 1) \cap [0, 10]$ and $(-1, 1) \in \tau$.

(ii) Consider $\mathbb{R}$ with the usual topology τ and $Y = \mathbb{Z}$. Then the relative topology on Y is the discrete topology. To see this, first notice that any singleton in Y is open since, for any $x \in \mathbb{Z}$, we have $\{x\} = (x - \frac{1}{2}, x + \frac{1}{2}) \cap \mathbb{Z}$ and $(x - \frac{1}{2}, x + \frac{1}{2}) \in \tau$. Then, for any $A \in \mathcal{P}(\mathbb{Z})$, we can write $A = \cup_{x\in A}\{x\}$ which is open since it is a union of open sets.

(iii) Let X be a nonempty set and equip it with the indiscrete topology. Then, for any nonempty subset $Y \subseteq X$, the relative topology on Y is the indiscrete topology on Y.

Exercise 4.1.5. Let (X, τ) be a topological space and let $Y_2 \subseteq Y_1 \subseteq X$. Let τ_2 be the relative topology on Y_2 as a subspace of (X, τ). Let τ_1 be the relative topology on Y_1 as a subspace of (X, τ), and let τ_0 be the relative topology on Y_2 as a subspace of (Y_1, τ_1). Prove that $\tau_0 = \tau_2$.

Remark 4.1.6. The above exercise tells us that in the situation where $Y_2 \subseteq Y_1 \subseteq X$, where (X, τ) is a topological space, we can view Y_2 as a subspace of Y_1 and view Y_2 as a subspace of X simultaneously, as the relative topology for Y_2 as a subspace of Y_1 and the relative topology for Y_2 as a subspace of X are the same.

If (X, τ) is a topological space and $Y \subseteq X$, then the open sets in Y are precisely the intersection of Y with the open sets in X (we define the relative topology this way). The next theorem examines situations where other topological notions are introduced to our subspace via intersection. Note that $\overline{A}^Y$ denotes the closure of A as a subset of Y with the sugspace topology whereas $\overline{A}^X$ denotes the closure of A as a subset of X with the original topology.

Theorem 4.1.7. *Let (X, τ) be a topological space and let $Y \subseteq X$. Then:*

(i) $C \subseteq Y$ is closed in Y if and only if $C = D \cap Y$ where D is closed in X,

(ii) if $A \subseteq Y$, then $\overline{A}^Y = \overline{A}^X \cap Y$, and

(iii) If $\mathcal{B}$ is a basis for X, then $\{B \cap Y | B \in \mathcal{B}\}$ is a basis for Y.

Proof. To prove (i), let $C \subseteq Y$ be closed in Y with the subspace topology. Then $Y \setminus C \in \tau|_Y$ and so, there exists $U \in \tau$ such that $Y \setminus C = U \cap Y$. Then $C = (X \setminus U) \cap Y$ and $X \setminus U$ is closed in X, so let $D = X \setminus U$.

To prove (ii), let $A \subseteq Y$. Then,

$$\begin{aligned}
\overline{A}^Y &= \cap\{C | \, C \text{ is closed in } Y \text{ and } C \supseteq A\} \\
&= \cap\{D \cap Y | \, D \text{ is closed in } X \text{ and } D \supseteq A\} \\
&= (\cap\{D | \, D \text{ is closed in } X \text{ and } D \supseteq A\}) \cap Y \\
&= \overline{A}^X \cap Y
\end{aligned}$$

To prove (iii), let $\mathcal{B}$ be a base for τ and let $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$. Let $y \in Y$. Then there exists $B \in \mathcal{B}$ such that $y \in B$. Then $y \in B \cap Y \in \mathcal{B}_Y$. Next, let $D_1, D_2 \in \mathcal{B}_Y$ and let $y \in D_1 \cap D_2$, where $y \in Y$. Then $D_1 = B_1 \cap Y$ and $D_2 = B_2 \cap Y$ for some $B_1, B_2 \in \mathcal{B}$. Hence, $y \in B_1 \cap B_2$. Since $\mathcal{B}$ is a base for τ, there exists $B_3 \in \mathcal{B}$ such that $y \in B_3 \subseteq B_1 \cap B_2$. And, since $y \in Y$, we have that

$$y \in B_3 \cap Y \subseteq B_1 \cap B_2 \cap Y = (B_1 \cap Y) \cap (B_2 \cap Y) = D_1 \cap D_2.$$

□

Example 4.1.8. Let X with relation $\leq$ be a totally ordered set with at least two elements. Then, as discussed in the last chapter, we can equip X with the order topology τ. Let $Y \subseteq X$ with at least two elements. Then Y is a totally ordered set with relation $\leq \cap (Y \times Y)$ (when we view $\leq$ as a subset of $X \times X$). Hence, we can equip Y with the order topology τ_0. Also, since Y is a subset of X, we can equip Y with the relative topology τ_1. One might then be tempted to believe that $\tau_0 = \tau_1$ but this is not always the case. Indeed, it is easy to see that $\tau_0 \subseteq \tau_1$ since a subbase for τ_0 is the collection of sets of the form $(a, \infty)_Y = \{y \in Y | a < y\}$ and $(-\infty, a)_y = \{y \in Y | \, y < a\}$ and we can write $(a, \infty)_Y = (a, \infty)_X \cap Y \in \tau_1$ and $(-\infty, a)_Y = (-\infty, a)_X \cap Y \in \tau$, where $(a, \infty)_X = \{x \in X | \, a < x\}$ and $(-\infty, a)_X = \{x \in X | \, x < a\}$. To see that τ_1 is not always a subset of τ_0, take $X = \mathbb{R}$, with the usual total ordering, $Y = [0, 1) \cup \{5\}$. Then, in the subspace topology $\{5\}$ is open, since $\{5\} = (4, 6) \cap Y$ but $\{5\}$ is not open with respect to the order topology since any open set containing 5, with respect to the order, would need to also contain points close to 1.

Theorem 4.1.9. *Let (X, τ) and (Y, σ) be topological spaces and let $f : (X, \tau) \to (Y, \sigma)$ be continuous. Let $A \subseteq X$. Then $f|_A : (A, \tau|_A) \to (Y, \sigma)$ is continuous.*

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be continuous and let $A \subseteq X$. Let $O \in \sigma$. Then

$$(f|_A)^{-1}(O) = f^{-1}(O) \cap A \in \tau|_A$$

since f is continuous and so $f|_A$ is continuous. □

Theorem 4.1.10. *Let (X, τ) and (Y, σ) be a topological spaces and suppose $X = A \cup B$ where $A, B \in \tau$. Let $f : (X, \tau) \to (Y, \sigma)$. If $f|_A$ and $f|_B$ are continuous, then f is continuous.*

Proof. Let $O \in \sigma$. Then $(f|_A)^{-1}(O) = f^{-1}(O) \cap A \in \tau|_A$ and $(f|_B)^{-1}(O) = f^{-1}(O) \cap B \in \tau|_B$. Hence, there exists $U_1, U_2 \in \tau$ such that $f^{-1}(O) \cap A = U_1 \cap A$ and $f^{-1}(O) \cap B = U_2 \cap B$. Since $A, B \in \tau$, we then have that $U_1 \cap A, U_2 \cap B \in \tau$. Then,

$$f^{-1}(O) = \big(f^{-1}(O) \cap A\big) \cup \big(f^{-1}(O) \cap B\big) = (U_1 \cap A) \cup (U_2 \cap B) \in \tau$$

since it is a union of elements from τ. □

Exercise 4.1.11. Prove the above theorem where, instead of assuming that $A, B \in \tau$, assume A and B are closed with respect to τ.

Theorem 4.1.12. *Let (X, τ) and (Y, σ) be topological spaces and suppose $Z \subseteq Y$. Let $f : X \to Z$. Then $f : (X, \tau) \to (Z, \sigma|_Z)$ is continuous if and only if $f : (X, \tau) \to (Y, \sigma)$ is continuous.*

Proof. For the forward direction, suppose $f : (X, \tau) \to (Z, \sigma|_Z)$ is continuous. Let $O \in \sigma$. Since $f(O) \subseteq Z$, we have that $f^{-1}(O) = f^{-1}(O \cap Z)$ and $O \cap Z \in \sigma|_Z$. Since $f : (X, \tau) \to (Z, \sigma|_Z)$ is continuous, we then have that $f^{-1}(O) = f^{-1}(O \cap Z) \in \tau$.

For the other direction, suppose $f : (X, \tau) \to (Y, \sigma)$ is continuous. Let $O \in \sigma|_Z$. Then $O = U \cap Z$ for some $U \in \sigma$. Since $f : (X, \tau) \to (Y, \sigma)$ is continuous, we have that $f^{-1}(U) \in \tau$. But, $f(X) \subseteq Z$ so $f^{-1}(O) = f^{-1}(U \cap Z) = f^{-1}(U) \in \tau$. □

Up to this point, we have had more of an analytic perspective and interpretation of topology. At this stage, we can discuss some of the geometric interpretations of topology.

Example 4.1.13. 1. Let τ be the usual topology on $\mathbb{R}$. Consider the function $f : ((0,1)\tau|_{(0,1)}) \to ((0,100), \tau|_{(0,100)})$ given by $f(x) = 100x$. Clearly f is a continuous bijection and f^{-1} is also a continuous bijection. Hence, $(0,1)$ and $(0,100)$, when equipped with the subspace topology inherited by $(\mathbb{R}, \tau)$ are homeomorphic. The geometric interpretation here is that topological spaces can be stretched and keep their topological structure. We could also replace the interval $(0,1)$ above with $[0,1]$ and the interval $(0,100)$ with $[0,100]$ to see that $[0,1]$ and $[0,100]$ are also homeomorphic.

2. Let τ be the usual topology on $\mathbb{R}$ and let σ be the usual topology on $\mathbb{R}^2$. Let $G = \{(x,x^2)|\, x \in [0,1]\}$ and define $f : ([0,1], \tau|_{[0,1]}) \to (G, \sigma|_G)$ by $f(x) = (x,x^2)$. Then f is a homeomorphism and so $[0,1]$ and G are homeomorphic when equipped with their respective subspace topologies inherited by the usual topologies. The geometric interpretation here is that we can bend topological spaces without changing the topological structure.

The above examples illustrate how topological spaces can be stretched and bent and maintain their topological structure. We tend to rely more on intuition when discussing such scenarios. For example, if we equip the unit sphere $S_1 = \{(x,y)|\, x^2 + y^2 = 1\}$ with the subspace topology inherited by $\mathbb{R}^2$ with the usual topology and we equip the unit square (just the surface, not the interior) with vertices $(-\frac{1}{2}, -\frac{1}{2})$, $(-\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{2}, -\frac{1}{2})$, and $(\frac{1}{2}, \frac{1}{2})$ with the subspace topology inherited by the usual topology on $\mathbb{R}^2$, then it is easy to see that they are homeomorphic since we can bend and stretch S_1 into the unit cube. We use intuition rather than proof primarily because, while it is clear these spaces are homeomorphic, providing a homeomorphism between them can be rather technical and tedious.

While we are allowed to stretch and bend topological spaces and maintain their structure, what we are not allowed to do is break them into pieces or glue parts together. For example, taking an interval such as $[0,1]$ or $[0,1)$ and trying to wrap it into a circle so that it is homeomorphic to S_1 will not work as we cannot glue the endpoints together. We also can't take an interval like $[0,1]$ and try to break it into two intervals such as $[0, \frac{1}{2}]$ and $(\frac{1}{2}, 1]$ or $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. The reasons for this will become clear later when we discuss properties of topological spaces which are preserved by homeomorphisms. At this time, it is rather difficult (but not impossible) to prove such spaces are not homeomorphic as we would be tasked with proving that it is not possible to define a homeomorphism between them.

Since we were studying the subspace topology here, we were very explicit about the subspace topologies being used and from which topologies they were inherited. In general, this can be rather tedious and so some collective understandings are in order. For example, if we mention that the interval $[0,1]$ has the usual topology, what we mean is that $[0,1]$ has the subspace topology it inherits from the usual topology on $\mathbb{R}$. Similarly, to say the unit square has the usual topology, we mean that it has the subspace topology it inherits from $\mathbb{R}^2$ with the usual topology.

4.2 The Product Topology

There are many ways one can define a topology on the product of topological spaces especially if it is an infinite product of metric spaces. We will begin the discussion by defining a topology on a finite product of topological spaces which has a very natural definition. Before we do so, let's look at the case when we take the product of $\mathbb{R}$, with the usual topology, with itself.

Let τ be the usual topology on $\mathbb{R}$. It's natural to then ask if we can define a topology τ_0 on $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ by making τ_0 the collection of all sets of the form $O_1 \times O_2$, where $O_1, O_2 \in \tau$. Unfortunately, this does not produce a topology on $\mathbb{R}^2$. For example, $(0,2) \times (0,2) \cup (0,1) \times (0,3) \notin \tau_0$. Thankfully, τ_0 forms a base for a topology on $\mathbb{R}^2$ which is precisely the usual topology on $\mathbb{R}^2$!

Proposition 4.2.1. *Let τ be the usual topology on $\mathbb{R}$ and define $\mathcal{B}$ to be the collection* *of all sets of the form $O_1 \times O_2$, where $O_1, O_2 \in \tau$. Then $\mathcal{B}$ is a base for a topology τ_1 on $\mathbb{R}^2$. Further, if τ_2 is the usual topology on $\mathbb{R}^2$, then $\tau_1 = \tau_2$.*

Proof. First, let's prove $\mathcal{B}$ is a base for a topology on $\mathbb{R}^2$. Let $(x, y) \in \mathbb{R}^2$. Then $(x, y) \in (x-1, x+1) \times (y-1, y+1) \in \mathcal{B}$. Now, let $O_1 \times O_2, U_1 \times U_2 \in \mathcal{B}$ and suppose $(x, y) \in (O_1 \times O_2) \cap (U_1 \times U_2)$. But,

$$(O_1 \times O_2) \cap (U_1 \times U_2) = (O_1 \cap U_1) \times (O_2 \cap U_2) \in \mathcal{B}.$$

Hence, $\mathcal{B}$ is a base for a topology τ_1 on $\mathbb{R}^2$.

To prove $\tau_1 = \tau_2$, first note that basic open sets with respect to τ_1 are open rectangles while basic open sets with respect to τ_2 are open disks. Given a basic open set O in τ_1, it is therefore an open rectangle. Given a point $x \in O$, we can find an open disk U containing x inside of O. Thus, we can write O as an arbitrary union of open disks and so $O \in \tau_2$. Hence, $\tau_1 \subseteq \tau_2$. Similarly, given a basic open set O in τ_2, it is an open disk and, for any $x \in O$, we can find an open rectangle containing x which is inside of O. Thus, we can write O as a union of open rectangles. Hence, $O \in \tau_1$. Therefore, $\tau_1 = \tau_2$. $\square$

Thus, for a finite product of topological spaces, it seems as though the natural way to define a topology is to take as a base all the products of open sets from their corresponding topologies. Of course, we still have to check that this will produce a topology on the product in this more general case. Further, this strategy also works for an infinite product of topological spaces so we will check this as well. Although, for infinite products, we typically do not define the topology this way. We'll discuss this a bit later.

Theorem 4.2.2. *Let (X_α, τ_α) be a topological space for all $\alpha \in I$, for some index set I. Let $\mathcal{B}$ be the collection of all sets of the form $\prod_{\alpha \in I} O_\alpha$, where $O_\alpha \in \tau_\alpha$. Then $\mathcal{B}$ forms a base for a topology on $\prod_{\alpha \in I} X_\alpha$.*

Proof. Let $f \in \prod_{\alpha \in I} X_\alpha$. Since $X_\alpha \in \tau_\alpha$ for all $\alpha \in I$, we have that $\prod_{\alpha \in I} X_\alpha \in \mathcal{B}$.

Let $\prod_{\alpha \in I} O_\alpha, \prod_{\alpha \in I} U_\alpha \in \mathcal{B}$ where $O_\alpha, U_\alpha \in \tau_\alpha$ for all $\alpha \in I$. Further, suppose $f \in \left(\prod_{\alpha \in I} O_\alpha\right) \cap \left(\prod_{\alpha \in I} U_\alpha\right)$. Then $f(\alpha) \in O_\alpha \cap U_\alpha$ for all $\alpha \in I$ and $O_\alpha \cap U_\alpha \in \tau_\alpha$ so $\prod_{\alpha \in I}(O_\alpha \cap U_\alpha) \in \mathcal{B}$ and

$$f(\alpha) \in \prod_{\alpha \in I}(O_\alpha \cap U_\alpha) \subseteq \left(\prod_{\alpha \in I} O_\alpha\right) \cap \left(\prod_{\alpha \in I} U_\alpha\right).$$

Hence, $\mathcal{B}$ is a base for a topology on $\prod_{\alpha \in I} X_\alpha$. □

Definition 4.2.3. Let (X_α, τ_α) be a topological space for all $\alpha \in I$, for some index set I. The topology generated by the base $\mathcal{B}$ given above is called the **box topology** on $\prod_{\alpha \in I} X_\alpha$.

Example 4.2.4. As we saw earlier, if $\mathbb{R}$ is equipped with the usual topology, then the box topology on $\mathbb{R}^2$ is precisely the usual topology on $\mathbb{R}^2$. Similarly, if $\mathbb{R}$ is equipped with the usual topology, then for any $n \in \mathbb{Z}_+$, the box topology on $\mathbb{R}^n$ is precisely the usual topology on $\mathbb{R}^n$.

Exercise 4.2.5. Let (X, τ) and (Y, σ) be topological spaces. Let A be a closed set in X and B be a closed set in Y. Prove that $A \times B$ is a closed set in $X \times Y$ when we equip $X \times Y$ with the box topology.

Note that we also have the following theorem which we will make use of later.

Theorem 4.2.6. *Let (X_α, τ_α) be a topological space for all $\alpha \in I$, for some index set I. Suppose $\mathcal{B}_\alpha$ is a base for τ_α, for all $\alpha \in I$. Let $\mathcal{B}$ be the collection of all sets of the form $\prod_{\alpha \in I} B_\alpha$, where $B_\alpha \in \mathcal{B}_\alpha$. Then $\mathcal{B}$ forms a base for the box topology on $\prod_{\alpha \in I} X_\alpha$.*

Proof. Let τ be the box topology on $\prod_{\alpha \in I} X_\alpha$. We first want to show that $\mathcal{B}$ forms a base for some topology on $\prod_{\alpha \in I} X_\alpha$. To do this, we have to check that $\mathcal{B}$ satisfies properties (i) and (ii) of Theorem **??**.

For property (i), let $f \in \prod_{\alpha \in I} X_\alpha$. Then $f(\alpha) \in X_\alpha$ and $\mathcal{B}_\alpha$ is a base for τ_α so there exists $B_\alpha \in \mathcal{B}_\alpha$ such that $f(\alpha) \in B_\alpha$. Then $f \in \prod_{\alpha \in I} B_\alpha \in \mathcal{B}$.

For property (ii), let $B_1, B_2 \in \mathcal{B}$ and let $f \in B_1 \cap B_2$. Well, $B_1 = \prod_{\alpha \in I} B_{\alpha,1}$ and $B_2 = \prod_{\alpha \in I} B_{\alpha,2}$ for some $B_{\alpha,1}, B_{\alpha,2} \in \mathcal{B}_\alpha$. Then, since $f \in B_1 \cap B_2$, we have that $f(\alpha) \in B_{\alpha,1} \cap B_{\alpha,2}$. Since $\mathcal{B}_\alpha$ is a base, there exists $B_{\alpha,3} \in \mathcal{B}_\alpha$ such that $f(\alpha) \in B_{\alpha,3} \subseteq B_{\alpha,1} \cap B_{\alpha,2}$. Then, $f \in \prod_{\alpha \in I} B_{\alpha,3} \subseteq B_1 \cap B_2$.

So, we now have that $\mathcal{B}$ is a base for a topology, say σ, on $\prod_{\alpha \in I} X_\alpha$. It is now left to show that $\sigma = \tau$. Since $\mathcal{B} \subseteq \tau$, we immediately obtain that $\sigma \subseteq \tau$. For the other inclusion, let O be a basic open set with respect to τ. Then $O = \prod_{\alpha \in I} O_\alpha$, where $O_\alpha \in \tau_\alpha$, for all $\alpha \in I$. Let $f \in O$. Then, $f(\alpha) \in O_\alpha$ and $\mathcal{B}_\alpha$ is a base for τ_α so there exists $B_{\alpha,f} \in \tau_\alpha$ such that $f(\alpha) \in B_{\alpha,f} \subseteq O_\alpha$. Then

$$f \in \prod_{\alpha \in I} B_{\alpha,f} \subseteq \prod_{\alpha \in I} O_\alpha = O$$

and $\prod_{\alpha \in I} B_{\alpha,f} \in \mathcal{B}$ so,

$$O = \bigcup_{f \in O} \left(\prod_{\alpha \in I} B_{\alpha,f} \right) \in \sigma.$$

Hence, $\tau = \sigma$ and so $\mathcal{B}$ is a base for τ. □

For infinite products of topological spaces, the box topology turns out to be too strong of a topology. That is, there are far too many open sets. For this reason, we often use a different topology, which we will now discuss.

Theorem 4.2.7. *Let (X_α, τ_α) be a topological space for all $\alpha \in I$, for some index set I. Define $\mathcal{B}$ to be the collection of all sets of the form $\prod_{\alpha \in I} O_\alpha$, where $O_\alpha \in \tau_\alpha$ and $O_\alpha = X_\alpha$ for all but a finite number of $\alpha \in I$. Then $\mathcal{B}$ forms a base for a topology on $\prod_{\alpha \in I} X_\alpha$.*

Proof. We have to show $\mathcal{B}$ satisfies properties (i) and (ii) of Theorem **??**. For property (i), let $f \in \prod_{\alpha \in I} X_\alpha$. Since $\prod_{\alpha \in I} X_\alpha \in \mathcal{B}$ we are done.

For property (ii), let $B_1, B_2 \in \mathcal{B}$ and let $f \in B_1 \cap B_2$. Well, $B_1 = \prod_{\alpha \in I} O_\alpha$ and $B_2 = \prod_{\alpha \in I} U_\alpha$ where $O_\alpha, U_\alpha \in \tau_\alpha$, $O_\alpha = X_\alpha$ for all but finitely many $\alpha \in I$, and $U_\alpha = X_\alpha$ for all but finitely many $\alpha \in I$. Since $f \in B_1 \cap B_2$, we have that $f(\alpha) \in O_\alpha \cap U_\alpha$ for all $\alpha \in I$. Note that there can only be finitely many α such that $O_\alpha \cap U_\alpha \neq X_\alpha$. For these α, let $V_\alpha = O_\alpha \cap U_\alpha \in \tau_\alpha$, since τ_α is a topology. For all other α, let $V_\alpha = X_\alpha$. Then $f \in \prod_{\alpha \in I} V_\alpha \subseteq B_1 \cap B_2$, where $\prod_{\alpha \in I} V_\alpha \in \mathcal{B}$. Thus, we have also checked property (ii) and so $\mathcal{B}$ forms a base for a topology on $\prod_{\alpha \in I} X_\alpha$. □

Definition 4.2.8. Let (X_α, τ_α) be a topological space for all $\alpha \in I$, for some index set I. Define $\mathcal{B}$ to be the collection of all sets of the form $\prod_{\alpha \in I} O_\alpha$, where $O_\alpha \in \tau_\alpha$ and $O_\alpha = X_\alpha$ for all but a finite number of $\alpha \in I$. Then the topology generated by $\mathcal{B}$ is called the **product topology** on $\prod_{\alpha \in I} X_\alpha$.

Note that if I is a finite set, then the product topology and the box topology on $\prod_{\alpha \in I} X_\alpha$ are equal. As with the box topology, if each topology in our product has a base, then we can use basic open sets to define a base for the product topology as the next proposition shows.

Proposition 4.2.9. *Let (X_α, τ_α) be a topological space for all $\alpha \in I$, for some index set I.Suppose $\mathcal{B}_\alpha$ is a base for τ_α, for all $\alpha \in I$. Define $\mathcal{B}$ to be the collection of all sets of the form $\prod_{\alpha \in I} B_\alpha$, where $O_\alpha \in \mathcal{B}_\alpha$ and $B_\alpha = X_\alpha$ for all but a finite number of $\alpha \in I$. Then $\mathcal{B}$ is a base for the product topology on $\prod_{\alpha \in I} X_\alpha$.*

Proof. The proof is a following exercise. □

Exercise 4.2.10. Prove Proposition 4.2.9. *Hint: The proof is very similar to the proof of Theorem 4.2.6. We just have to make sure that when we create $B \in \mathcal{B}$, the αth coordinate is not equal to X_α for only finitely many α.*

It is often easier to use projections to describe open sets in the product topology. Let us first define a projection in this setting.

Definition 4.2.11. Let (X_α, τ_α) be a topological space for all $\alpha \in I$, for some index set I. For each $\gamma \in I$, define $\pi_\gamma : \prod_{\alpha \in I} X_\alpha \to X_\gamma$ by $\pi_\gamma(f) = f(\gamma)$ where we are defining the elements $f \in \prod_{\alpha \in I} X_\alpha$ as functions $f : I \to \prod_{\alpha \in I} X_\alpha$ where $f(\alpha) \in X_\alpha$ for all $\alpha \in I$. The map π_γ is the **projection map** from $\prod_{\alpha \in I} X_\alpha$ to X_γ or simply the γ-**projection map**.

Example 4.2.12. Perhaps some examples would help to clarify the situation.

(i) Suppose $I = \mathbb{Z}_+$ and $X_\alpha = \mathbb{R}$, for all $\alpha \in I$. Then the elements of $\prod_{\alpha \in I} X_\alpha$ are precisely all the real-valued sequences indexed by $\mathbb{Z}_+$. Take, for example, the sequence $f = (1, \frac{1}{4}, \frac{1}{9}, \ldots)$. Then $\pi_2(f) = \frac{1}{4}$ while $\pi_{10}(f) = \frac{1}{100}$.

(ii) Suppose instead that $I = \mathbb{R}$ and $X_\alpha = \mathbb{R}$, for all $\alpha \in I$. Then the elements of $\prod_{\alpha \in I} X_\alpha$ are precisely all the functions $f : \mathbb{R} \to \mathbb{R}$. As an example, let $f \in \prod_{\alpha \in I} X_\alpha$ be given by $f(x) = x^2$. Then $\pi_2(f) = f(2) = 4$ while $\pi_{\sqrt{3}}(f) = f(\sqrt{3}) = 3$.

With projections, we now have another way to describe basic open sets in the product topology. If (X_α, τ_α) are topological spaces and $\mathcal{B}$ is defined as in Defiinition 4.2.8 so that it is a base for the product topology on $\prod_{\alpha \in I} X_\alpha$, then any $U \in \mathcal{B}$ can be written as

$$U = \pi_{\alpha_1}^{-1}(O_{\alpha_1}) \cap \pi_{\alpha_2}^{-1}(O_{\alpha_2}) \cap \cdots \cap \pi_{\alpha_n}^{-1}(O_{\alpha_n})$$

for some $n \in \mathbb{Z}_+$ and $O_{\alpha_k} \in \tau_{\alpha_k}$, for $k = 1, 2, \ldots, n$. If $\mathcal{B}_\alpha$ is a base for τ_α, for all $\alpha \in I$, we can instead require above that each $O_{\alpha_k} \in \mathcal{B}_{\alpha_k}$ instead of τ_{α_k}, for $k = 1, 2, \ldots, n$. With this interpretation of the elements of $\mathcal{B}$ it is easy to see that each projection map is continuous. The next theorem clarifies this but also states that the product topology is the weakest topology $\prod_{\alpha \in I} X_\alpha$ which makes all of the projection maps continuous.

Theorem 4.2.13. *Let (X_α, τ_α) be a topological space for each $\alpha \in I$, where I is some index set. Let τ be the product topology on $\prod_{\alpha \in I} X_\alpha$. Then, for each $\gamma \in I$, the* *projection map* $\pi_\gamma : \prod_{\alpha \in I} X_\alpha \to X_\gamma$ *is continuous. Further, if* τ_0 *is another topology on* $\prod_{\alpha \in I} X_\alpha$ *for which each projection map* $\pi_\gamma : \prod_{\alpha \in I} X_\alpha \to X_\gamma$ *is continuous, then* $\tau \subseteq \tau_0$.

Proof. From the discussion immediately preceding the statement of the theorem, it is clear that π_γ is continuous for each $\gamma \in I$. Thus, it remains to prove that the product topology, τ, is the weakest topology which makes all of the projection maps continuous. Let τ_0 be another topology on $\prod_{\alpha \in I} X_\alpha$ such that π_γ is continuous for all $\gamma \in I$. Let O be a basic open set in τ. Then there exists $n \in \mathbb{Z}_+$ and $\alpha_1, \alpha_2, \ldots, \alpha_n \in I$ such that

$$O = \pi_{\alpha_1}^{-1}(O_{\alpha_1}) \cap \pi_{\alpha_2}^{-1}(O_{\alpha_2}) \cap \cdots \cap \pi_{\alpha_n}^{-1}(O_{\alpha_n})$$

where $O_{\alpha_k} \in \tau_{\alpha_k}$ for all $k = 1, 2, \ldots, n$. Since π_{α_k} is continuous with respect to τ_0, we have that $\pi_{\alpha_k}^{-1}(O_{\alpha_k}) \in \tau_0$ for all $k = 1, 2, \ldots, n$. Thus,

$$O = \pi_{\alpha_1}^{-1}(O_{\alpha_1}) \cap \pi_{\alpha_2}^{-1}(O_{\alpha_2}) \cap \cdots \cap \pi_{\alpha_n}^{-1}(O_{\alpha_n}) \in \tau_0$$

since it is a finite intersection of elements from τ_0. Hence, $\tau \subseteq \tau_0$ and so τ is weaker than τ_0. □

The next theorem gives us a way to prove that a function from a topological space into a space equipped with the product topology is continuous by simply checking whether the projection maps composed with the function are continuous.

Theorem 4.2.14. *Let* (X, τ) *be a topological space and let* $(Y_\alpha, \sigma_\alpha)$ *be a topological space for each* $\alpha \in I$*, where* I *is some index set. Let* σ *be the product topology on* $\prod_{\alpha \in I} Y_\alpha$ *and let* $f : (X, \tau) \to (\prod_\alpha Y_\alpha, \sigma)$*. If* $\pi_\alpha \circ f : (X, \tau) \to (Y_\alpha, \sigma_\alpha)$ *is continuous, for all* $\alpha \in I$*, then* $f : (X, \tau) \to (\prod_\alpha Y_\alpha, \sigma)$ *is continuous.*

Proof. Suppose $\pi_\alpha \circ f : (X, \tau) \to (Y_\alpha, \tau_\alpha)$ is continuous, for all $\alpha \in I$. Let O be a basic open set in σ. Then there exists $n \in Z_+$, $\alpha_1, \alpha_2, \ldots, \alpha_n \in I$, and $O_{\alpha_k} \in \sigma_{\alpha_k}$ such that

$$O = \pi_{\alpha_1}^{-1}(O_{\alpha_1}) \cap \pi_{\alpha_2}^{-1}(O_{\alpha_2}) \cap \cdots \cap \pi_{\alpha_n}^{-1}(O_{\alpha_n}).$$

Since $\pi_{\alpha_k} \circ f$ is continuous, we have that $(\pi_{\alpha_k} \circ f)^{-1}(O_{\alpha_k}) \in \tau$ for all $k = 1, 2, \ldots, n$. Then,

$$\begin{aligned} f^{-1}(O) &= f^{-1}\left(\pi_{\alpha_1}^{-1}(O_{\alpha_1}) \cap \pi_{\alpha_2}^{-1}(O_{\alpha_2}) \cap \cdots \cap \pi_{\alpha_n}^{-1}(O_{\alpha_n})\right) \\ &= f^{-1}\left(\pi_{\alpha_1}^{-1}(O_{\alpha_1})\right) \cap f^{-1}\left(\pi_{\alpha_2}^{-1}(O_{\alpha_2})\right) \cap \cdots \cap f^{1-}\left(\pi_{\alpha_n}^{-1}(O_{\alpha_n})\right) \\ &= (\pi_{\alpha_1} \circ f)^{-1}(O_{\alpha_1}) \cap (\pi_{\alpha_2} \circ f)^{-1}(O_{\alpha_2}) \cap \cdots \cap (\pi_{\alpha_n} \circ f)^{-1}(O_{\alpha_n}) \\ &\in \tau \end{aligned}$$

since it is a finite intersection of sets from τ. Thus, f is continuous. □

An application of the above theorem is the following theorem.

Theorem 4.2.15. *Let (X, τ) be a topological space and let $f : (X, \tau) \to \mathbb{R}$ and $g : (X, \tau) \to \mathbb{R}$, where $\mathbb{R}$ is equipped with the usual topology. If f and g are continuous, then $f + g$ is continuous.*

Proof. Define $h : (X, \tau) \to \mathbb{R} \times \mathbb{R}$ by $h(x) = (f(x), g(x))$, where $\mathbb{R} \times \mathbb{R}$ has the product topology (which is equal to the usual topology). Let π_1 be the projection map onto the first coordinate of $\mathbb{R} \times \mathbb{R}$ and π_2 be the projection map onto the second coordinate. Then $\pi_1 \circ h = f$, which is continuous, and $\pi_2 \circ h = g$, which is continuous. Hence, by Theorem 4.2.15, the function h is continuous. Now, consider the function $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ where $+(a, b) = a + b$. While it looks a bit different, the fact that $+$ is continuous is equivalent to the fact the sum of two convergent sequences converges to the sum of their limits from calculus class. If this is unsatisfactory, let $B(z, r)$ be an open ball in $\mathbb{R}$, for some $x \in \mathbb{R}$ and $r > 0$. Then $+^{-1}(B(z, r))$ is the open region in the xy-plane strictly above the line $y = -x + z - r$ and strictly below the line

$y = -x + z + r$, which is clearly an open set. Then, by Exercise **??**, the function $f + g = + \circ h$ is continuous. □

Corollary 4.2.16. *Let $n \in \mathbb{Z}_+$, $\alpha_1, \alpha_2, \ldots, \alpha_2 \in \mathbb{R}$, and $f_i : (X, \tau) \to \mathbb{R}$ be continuous for each $i = 1, 2, \ldots, n$, where (X, τ) is a topological space and $\mathbb{R}$ is equipped with the usual topology. Then $\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n$ is continuous.*

Proof. It suffices to prove that if $f : (X, \tau) \to \mathbb{R}$ is continuous and $\alpha \in \mathbb{R}$, then αf is continuous, where $(\alpha f)(x) = \alpha f(x)$, for all $x \in X$. Once this is complete, we can simply appeal to Theorem 4.2.15. If we define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = \alpha x$, then g is obviously continuous as g is just a line from our high school classes. If we would rather use the ϵ-δ definition of continuous functions, then, for any $\epsilon > 0$, simply let $\delta = \frac{\epsilon}{|\alpha|+1}$. Then, we have that $\alpha f = g \circ f$ and so, by Exercise **??**, the function αf is continuous. □

Definition 4.2.17. Let (X, τ) be a topological space. We say a set $A \subseteq X$ is **dense** in X if $\overline{A} = X$.

If you have taken a course in real analysis, then you're aware that $\overline{\mathbb{Q}} = \mathbb{R}$. Hence, we would say $\mathbb{Q}$ is dense in $\mathbb{R}$. The next exercise gives us a useful way to check if a subset is dense.

Exercise 4.2.18. Let (X, τ) be a topological space and let A be a subset of X. The set A is dense in X if and only if, for every nonempty $O \in \tau$, we have $A \cap O \neq \emptyset$.

Theorem 4.2.19. *Let (X, τ) be a topological space and let σ be the usual topology on $\mathbb{R}$. Suppose $f : (X, \tau) \to (\mathbb{R}, \sigma)$ and $g : (X, \tau) \to (\mathbb{R}, \sigma)$ are continuous. Let $A = \{x \in X | \, f(x) = g(x)\}$. If A is dense in X, then $f(x) = g(x)$ for all $x \in X$.*

Proof. The reader is walked through the proof in the following exercise. □

Exercise 4.2.20. Prove Theorem 4.2.19. *Hint: Let $h : (X, \tau) \to (\mathbb{R}, \sigma)$, where $h(x) = f(x) - g(x)$. From Corollary 4.2.16, we know h is continuous. Now, use the* *fact that h is continuous to prove A is closed. Now, since A is closed, what does that tell us about the relationship between A and $\overline{A}$? You should now be able to conclude that $A = X$ which means $f(x) = g(x)$ for all $x \in X$.*

The above theorem tells us that for any continuous function f from a general topological space (X, τ) into $\mathbb{R}$, with the usual topology, knowing what f does on a dense subset of X completely determines what f does on all of X. This fact has a variety of applications. Suppose, for example, that we have a continuous function f we are searching for but all we know is what it does on a dense subset of X. If we can find a continuous function g on X which agrees with f on that dense subset, then we must have that $f = g$.

Suppose instead, we are in a setting where we have the function f on hand (suppose it's a very complicated function with no nice rule to tell us where to send each element of the domain) but it would be very tedious to send a full description of the function to someone else (perhaps over the internet with multiple levels of security). We could, instead, describe what the function does on any dense subset of the domain, of our choosing, and have the person on the other side find a function g agreeing with f on that dense subset (assuming we have an efficient way to do so). The person on the other side then knows the function g that they found is actually f!

As was mentioned at the end of Chapter 2, there is no metric we can define on $\mathbb{R}^{\mathbb{R}}$ so that a sequence $(f_n)_{n=1}^{\infty}$ in $\mathbb{R}^{\mathbb{R}}$ converges with respect to the metric if and only if the sequence converges pointwise (although we still have not seen why). There is, however, a topology we can define on $\mathbb{R}^{\mathbb{R}}$ which does this and it is precisely the product topology, as the next example shows.

Example 4.2.21. For the sake of clarity, in a real analysis course, we say a sequence of functions $(f_n)_{n=1}^{\infty}$ **converges pointwise** to a function f if, for all $x \in \mathbb{R}$, $f_n(x) \to f(x)$. To be precise, $f_n \to f$ pointwise if, for all $x \in \mathbb{R}$ and all $\epsilon > 0$, there exists $N \in \mathbb{Z}_+$ such that for all $n \geq N$, we have that $|f_n(x) - f(x)| < \epsilon$. Now, let $(f_n)_{n=1}^{\infty}$

be a sequence in $\mathbb{R}^{\mathbb{R}}$ equipped with the product topology τ and $f \in \mathbb{R}^{\mathbb{R}}$. We claim that $f_n \xrightarrow{\tau} f$ if and only if $f_n \to f$ pointwise.

To see this, first suppose $f_n \xrightarrow{\tau} f$. Let $x \in \mathbb{R}$ and $\epsilon > 0$. Let d denote the usual metric on $\mathbb{R}$. Then $U = \pi_x^{-1}(B_d(f(x), \epsilon))$ is a basic open set in the product topology so there exists $N \in \mathbb{Z}_+$ such that, for all $n \geq N$, we have that $f_n \in \pi_x^{-1}(B_d(f(x), \epsilon))$. That is, $\pi_x(f_n) \in B_d(f(x), \epsilon)$, and so, $|f_n(x) - f(x)| < \epsilon$. Hence, $f_n(x) \to f(x)$. This holds for all $x \in \mathbb{R}$ and so $f_n \to f$ pointwise.

Now, suppose $f_n \to f$ pointwise. Let U be a basic open neighborhood of f. Then there exists $k \in \mathbb{Z}_+$ and $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k \in \mathbb{R}$ as well as $\epsilon_1, \epsilon_2, \ldots, \epsilon_k \in (0, \infty)$ such that

$$U = \pi_{x_1}^{-1}(B_d(y_1, \epsilon_1)) \cap \pi_{x_2}^{-1}(B_d(y_2, \epsilon_2)) \cap \cdots \cap \pi_{x_k}^{-1}(B_d(y_k, \epsilon_k)). \tag{4.1}$$

Since $f \in U$, we know that $|f(x_i) - y_i| < \epsilon_i$, for all $i = 1, 2, \ldots, k$. Now, for $i = 1, 2, \ldots, k$, let $\gamma_i = \epsilon_i - |f(x_i) - y_i|$ and define

$$U_0 = \pi_{x_1}^{-1}(B_d(f(x_1), \gamma_1)) \cap \pi_{x_2}^{-1}(B_d(f(x_2), \gamma_2)) \cap \cdots \cap \pi_{x_k}^{-1}(B_d(f(x_k), \gamma_k)).$$

Then $f \in U_0 \subseteq U$. For each $i = 1, 2, \ldots, k$, since $f_n \to f$ pointwise, there exists $N_i \in \mathbb{Z}_+$ such that, for all $n \geq N_i$, we have that $|f_n(x_i) - f(x_i)| < \gamma_i$. Let $N = \max\{N_1, N_2, \ldots, N_k\}$. Then, for $n \geq N$, we have that $|f_n(x_i) - f(x_i)| < \gamma_i$, for all $i = 1, 2, \ldots, k$ and so $f_n \in U_0 \subseteq U$. Hence, $f_n \xrightarrow{\tau} f$.

If we equip $\mathbb{R}^{\mathbb{R}}$ with the box topology τ rather than the product topology, then pointwise convergence does not imply convergence with respect to τ. It is worth reexamining where the proof above breaks down. Note that if we are using the box topology, then in Equation (4.1), we would have to take the set U to be an infinite intersection of sets. If we then look at how we defined N later on, we could not define N to be the maximum of all the N_i as there would now be infinitely many of them.

This is one argument as to why we say, for infinite products, the box topology is "too strong." Having coordinate-wise convergence does not guarantee overall convergence.

4.3 The Weak Topology

We saw in the last section that the product topology τ defined on the product of topological spaces (X_α, τ_α), $\alpha \in I$, is the weakest topology which makes all of the projection maps continuous. We also saw that we can construct a base for the product topology with sets of the form

$$U = \pi_{\alpha_1}^{-1}(O_{\alpha_1}) \cap \pi_{\alpha_2}^{-1}(O_{\alpha_2}) \cap \cdots \cap \pi_{\alpha_n}^{-1}(O_{\alpha_n}).$$

for $n \in \mathbb{Z}_+$ and $O_{\alpha_i} \in \tau_{\alpha_i}$ for $\alpha_1, \ldots, \alpha_n \in I$. Hence, the sets of the form $\pi_\alpha^{-1}(O_\alpha)$ form a subbase for the product topology. So, in retrospect, we could have simply defined the product topology to be the topology generated by this subbase.

There is nothing preventing us from generalizing this situation to other functions and other topological spaces besides the product spaces. This is the idea behind the weak topology.

Definition 4.3.1. Let X be a set, I be an index set, and (X_α, τ_α) be topological spaces for all $\alpha \in I$. Further, suppose $f_\alpha : X \to X_\alpha$ for all $\alpha \in I$. Let $\mathcal{F} = \{f_\alpha | \, \alpha \in I\}$ and let $\mathcal{C} = \{f_\alpha^{-1}(O_\alpha) | \, \alpha \in I \text{ and } O_\alpha \in \tau_\alpha\}$. Then $\mathcal{C}$ is a subbase for a topology on X called the **weak topology** on X generated by $\mathcal{F}$. By construction, it is the weakest topology on X which makes all the functions in $\mathcal{F}$ continuous.

Quite often, the topological spaces (X_α, τ_α) are all the same topological space, in which case, the definition above isn't quite so technical looking. For example, say we have a set X and a collection of functions $\mathcal{F}$, where each $f \in \mathcal{F}$ is of the form $f : X \to \mathbb{R}$, where $\mathbb{R}$ is equipped with the usual topology τ. Then the weak topology on X is the topology with subbase $\{f^{-1}(O) : O \in \tau \text{ and } f \in \mathcal{F}\}$ and it is the weakest topology which makes all the functions in $\mathcal{F}$ continuous. The following example illustrates this scenario.

Example 4.3.2. Let $X = C([0,1])$, where $C([0,1])$ is the set of continuous real-valued functions on $[0,1]$. Recall that $C([0,1]) \subset \mathbb{R}^{[0,1]}$. Let $I = \{[a,b] |\, 0 \leq a < b \leq 1\}$. Then, for each $[a,b] \in I$, define $e_{[a,b]} : X \to \mathbb{R}$ by $e_{[a,b]}(f) = \int_a^b f(t)dt$, for all $f \in X$, where we equip $\mathbb{R}$ with the usual topology. Now, let $\mathcal{F} = \{e_{[a,b]} |\, [a,b] \in I\}$. Then $\mathcal{F}$ generates a topology on X and it is the weakest topology which makes every element of $\mathcal{F}$ continuous.

If (X,τ) and (Y,σ) are topological spaces and σ is the weak topology induced by a collection of maps $\mathcal{F} = \{f_\alpha | \alpha \in I\}$, where $f_\alpha : Y \to (Z_\alpha, \eta_\alpha)$, where (Z_α, η_α) is a topological space for each $\alpha \in I$, then the next theorem tells us how to check if a function $f : (X,\tau) \to (Y,\sigma)$ is continuous.

Theorem 4.3.3. *Let (X,τ) and (Y,σ) be topological spaces and σ be the weak topology induced by a collection of maps $\mathcal{F} = \{f_\alpha | \alpha \in I\}$, where $f_\alpha : Y \to (Z_\alpha, \eta_\alpha)$, where (Z_α, η_α) is a topological space for each $\alpha \in I$. We then have that $f : (X,\tau) \to (Y,\sigma)$ is continuous if and only if $f_\alpha \circ f : (X,\tau) \to (Z_\alpha, \eta_\alpha)$ is continuous.*

Proof. Let f be continuous and let $\alpha \in I$. Since σ is the weakest topology making each f_α continuous, f_α is continuous. Hence, $f_\alpha \circ f$ is continuous.

For the other direction, suppose $f_\alpha \circ f : (X,\tau) \to (Z_\alpha, \eta_\alpha)$ is continuous, for all $\alpha \in I$. Let O be a basic open set in Y. Then there exists $n \in \mathbb{Z}_+$, $\alpha_1, \alpha_2, \alpha_n \in I$ and $O_{\alpha_k} \in \eta_{\alpha_k}$ for $k = 1, 2, \ldots, n$, such that

$$O = f_{\alpha_1}^{-1}(O_{\alpha_1}) \cap f_{\alpha_2}^{-1}(O_{\alpha_2}) \cap \cdots \cap f_{\alpha_n}^{-1}(O_{\alpha_n}).$$

For each $k = 1, 2, \ldots, n$, since $O_{\alpha_k} \in \eta_{\alpha_k}$ and $f_{\alpha_k} \circ f$ is continuous, we have that

$(f_{\alpha_k} \circ f)^{-1}(O_{\alpha_k}) \in \tau$. Then,

$$\begin{aligned} f^{-1}(O) &= f^{-1}\left(f_{\alpha_1}^{-1}(O_{\alpha_1}) \cap f_{\alpha_2}^{-1}(O_{\alpha_2}) \cap \cdots \cap f_{\alpha_n}^{-1}(O_{\alpha_n})\right) \\ &= (f_{\alpha_1} \circ f)^{-1}(O_{\alpha_1}) \cap (f_{\alpha_2} \circ f)^{-1}(O_{\alpha_2}) \cap \cdots \cap (f_{\alpha_n} \circ f)^{-1}(O_{\alpha_n}) \\ &\in \tau \end{aligned}$$

since it is a finite intersection of elements from τ. Hence, f is continuous. □

If we are again in the scenario where all of the topological spaces (Z_α, η_α) are the same, say $Z_\alpha = \mathbb{R}$ and $\eta_\alpha = \eta$, where η is the usual topology on $\mathbb{R}$. Then, if we have a collection of functions $\mathcal{F} = \{f_\alpha : Y \to (\mathbb{R}, \eta) : \alpha \in I\}$ and we equip Y with the weak topology induced by $\mathcal{F}$, to check if a function $f : (X, \tau) \to (Y, \sigma)$ is continuous, we simply have to check if $f_\alpha \circ f : (X, \tau) \to (\mathbb{R}, \eta)$ is continuous for all $\alpha \in I$.

If the set $\mathcal{F}$ of functions we are using to generate a weak topology is small, then it will produce a weaker topology than a larger set of functions. To be precise, if $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then the topology induced by $\mathcal{F}_1$ will be weaker than the topology induced by $\mathcal{F}_2$ (checking this fact is a following exercise). Hence, we have to be careful about what we include and exclude from $\mathcal{F}$. We don't want to include so many functions that we produce too strong of a topology but we also don't want to exclude so many so that the topology does not have desirable properties. One way to ensure the topology generated has some of the properties we would like it to have is to require that it seperates points. We will see in a later section the properties this requirement produces. We will define what we mean by $\mathcal{F}$ separating points after we give the mentioned exercise.

Exercise 4.3.4. Let X be a set and let (Y, σ) be a topological space. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be collections of functions $f : X \to (Y, \sigma)$. Let τ_1 be the weak topology on X induced by $\mathcal{F}_1$ and τ_2 be the weak topology on X induced by $\mathcal{F}_2$. Prove that if $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then $\tau_1 \subseteq \tau_2$.

Definition 4.3.5. Let $\mathcal{F} = \{f_\alpha : X \to Y_\alpha | \, \alpha \in I\}$. We say $\mathcal{F}$ **seperates points** in X if, for any $x_1, x_2 \in X$, where $x_1 \neq x_2$, there exists $\alpha \in I$ such that $f_\alpha(x_1) \neq f_\alpha(x_2)$.

Roughly speaking, a collection of functions $\mathcal{F}$ seperates points in X if, given $x_1, x_2 \in X$ with $x_1 \neq x_2$, at least one member of $\mathcal{F}$ recognizes that x_1 and x_2 are distinct elements of X. One consequence of $\mathcal{F}$ **not** seperating points can be seen in the following exercise.

Exercise 4.3.6. Let X be a set and let $\mathcal{F} = \{f_\alpha : X \to (Y, \sigma_\alpha) : \alpha \in I\}$ and equip X with the weak topology τ induced by $\mathcal{F}$. Suppose $\mathcal{F}$ does not seperate points. That is, there exists $x_1, x_2 \in X$ such that $f_\alpha(x_1) = f_\alpha(x_2)$ for all $\alpha \in I$. Prove that a set U is an open neighborhood of x_1 if and only if U is an open neighborhood of x_2.

The above Exercise shows that if $\mathcal{F}$ does not recognize distinct points, then the topology it generates will not recognize distinct points either.

Example 4.3.7. (i) Let (X, d) be a metric space and, for each $x \in X$, define $f_x : X \to \mathbb{R}$ by $f_x(y) = d(x, y)$, for all $y \in X$. Now, let $\mathcal{F} = \{f_x | \, x \in X\}$. Then the topology induced by $\mathcal{F}$ on X is precisely the topology on X induced by the metric d. Further, $\mathcal{F}$ seperates points since, given $y_1, y_2 \in X$, where $y_1 \neq y_2$, we have that $f_{y_1}(y_1) = 0$, while $f_{y_1}(y_2) \neq 0$.

(ii) Consider the set $X = \mathbb{R}^{\mathbb{Z}_+}$ which, if you recall, is the set of all real-valued sequences $(x_n)_{n=1}^\infty$. Define $e_k : X \to \mathbb{R}$, where $\mathbb{R}$ has the usual topology, by $e_k((x_n)_{n=1}^\infty) = x_k$, for all $(x_n)_{n=1}^\infty \in X$. Now, let $\mathcal{F} = \{e_k | k \in \mathbb{Z}_+\}$. Then $\mathcal{F}$ generates a topology on X. Further, $\mathcal{F}$ seperates points. To see this, suppose we have two sequences $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ in X which are not equal. Then there must exist $m \in \mathbb{Z}_+$ such that $x_m \neq y_m$. Hence, $e_m((x_n)_{n=1}^\infty) \neq e_m((y_n)_{n=1}^\infty)$.

(iii) In Example 4.3.2 from earlier, the set $\mathcal{F}$ given there would seperate points although a proof of this would require some knowledge of analysis which we do not assume in these notes. If we modify Example 4.3.2 and suppose instead that X is the set of Riemann integrable real-valued functions on $[0,1]$, then we still have that $X \subset \mathbb{R}^{[0,1]}$. Let $I = \{[a,b] \mid 0 \leq a < b \leq 1\}$. Then, for each $[a,b] \in I$, define $e_{[a,b]} : X \to \mathbb{R}$ by $e_{[a,b]}(f) = \int_a^b f(t)dt$, for all $f \in X$. Now, let $\mathcal{F} = \{e_{[a,b]} \mid [a,b] \in I\}$. Then $\mathcal{F}$ generates a topology on X which does not seperate points. To see this, simply consider the functions f_1 and f_2 on $[0,1]$, where $f_1(\frac{1}{2}) = 1$ and $f_1(x) = 0$, if $x \neq \frac{1}{2}$, while $f_2(\frac{1}{2}) = 2$ and $f_2(x) = 0$, if $x \neq \frac{1}{2}$. Then $f_1 \neq f_2$ but $e_{[a,b]}(f_1) = e_{[a,b]}(f_2)$ for all $[a,b] \in I$.

Exercise 4.3.8. (i) Let $X = \mathbb{R}^{\mathbb{R}}$ and, for each $x \in X$, define $e_x : X \to \mathbb{R}$ by $e_x(f) = f(x)$. Let $\mathcal{F} = \{e_x \mid x \in \mathbb{R}\}$. Prove that $\mathcal{F}$ seperates points of X.

(ii) Let $X = C(\mathbb{R})$ and, for each $r \in \mathbb{Q}$, define $e_r : X \to \mathbb{R}$ by $e_r(f) = f(r)$. Let $\mathcal{F} = \{e_r \mid r \in \mathbb{Q}\}$. Prove that $\mathcal{F}$ seperates points of X. *Hint: Use Theorem 4.2.19 and the fact that* $\overline{\mathbb{Q}} = \mathbb{R}$.

4.4 The Quotient Topology

If we have maps from a set to some topological spaces, then the weak topology gives us a way to use the topologies of the codomains of the functions to define a topology on the domain. The quotient topology does the opposite. Given an onto map $f : X \to Y$, the quotient topology gives us a way to define a topology on Y using the topology on X.

Definition 4.4.1. Let (X, τ) be a topological space and let Y be a set. Further, suppose $f : (X, \tau) \to Y$ is onto. Let

$$\tau_f = \{O \subseteq Y \mid f^{-1}(O) \in \tau\}.$$

Then τ_f is a topology on Y called the **quotient topology** induced on Y by f. The topological space (Y, τ_f) is called a **quotient space** of X and the function f is called a **quotient map**.

Of course, we still have to verify that, indeed, τ_f is actually a topology on Y, which we do in the next proposition.

Proposition 4.4.2. *Let (X, τ) be a topological space and let Y be a set. Further, suppose $f : (X, \tau) \to Y$ is onto. Let*

$$\tau_f = \{O \subseteq Y | \, f^{-1}(O) \in \tau\}.$$

Then τ_f is a topology on Y. Further, τ_f is the strongest topology which makes f continuous. That is, if σ is a topology on Y and $f : (X, \tau) \to (Y, \sigma)$ is continuous, then $\sigma \subseteq \tau_f$.

Proof. First, we want to prove τ_f is a topology on Y. Well, $\emptyset \in \tau_f$ since $f^{-1}(\emptyset) = \emptyset \in \tau$ and $Y \in \tau_f$ since $f^{-1}(Y) = X \in \tau$. Now, let $O_i \in \tau_f$ for all $i \in I$. Then $f^{-1}(O_i) \in \tau$, for all $i \in I$. Then we have that

$$f^{-1}\left(\bigcup_{i \in I} O_i\right) = \bigcup_{i \in I} f^{-1}(O_i) \in \tau$$

and so $\cup_{i \in I} O_i \in \tau_f$. Hence, τ_f is closed under arbitrary unions. Next, let $n \in \mathbb{Z}_+$ and let $O_1, O_2, \ldots, O_n \in \tau_f$. Then $f^{-1}(O_1), f^{-1}(O_2), \ldots, f^{-1}(O_n) \in \tau$ and so

$$f^{-1}\left(\bigcap_{i=1}^{n} O_i\right) = \bigcap_{i=1}^{n} f^{-1}(O_i) \in \tau$$

and so $\cap_{i=1}^{n} O_i \in \tau_f$. Hence, τ_f is closed under finite intersections and thus, τ_f is a topology on Y.

Now, to show τ_f is the strongest topology on Y which makes f continuous, suppose σ is a topology on Y such that $f : (X, \tau) \to (Y, \sigma)$ is continuous. Let $O \in \sigma$. Since

$f : (X, \tau) \to (Y, \sigma)$ is cointinuous, we have that $f^{-1}(O) \in \tau$. This then implies that $O \in \tau_f$. Hence, $\sigma \subseteq \tau_f$ and so τ_f is stronger than σ. $\square$

At this stage, the reader might be wondering why we call (Y, τ_f) a "quotient space" and τ_f the "quotient topology" as this probably makes the reader think of quotient spaces in an abstract algebra class or, possibly, quotient spaces in analysis. The reason for this is important so we will explain it in detail.

Suppose (X, τ) is a topological space, Y is a set, and $f : (X, \tau) \to Y$ is onto. Now, define the quotient topology τ_f on Y. Next, define a relation $\sim_f$ on X by $x_1 \sim_f x_2 \Leftrightarrow f(x_1) = f(x_2)$. Then $\sim_f$ is an equivalence relation on X (checking this is a following exercise). Since $\sim_f$ is an equivalence relation, we can discuss the equivalence classes for $\sim_f$ of X. Let ${}^X\!/\!{}_{\sim_f}$ be the set of all equivalence classes of X. That is, ${}^X\!/\!{}_{\sim_f} = \{[x] | \, x \in X\}$ and recall that the equivalence classes of X partition X into disjoint sets (that is, the equivalence classes are disjoint and the union of all equivalence classes equals X). Next, define a topology ${}^\tau\!/\!{}_{\sim_f}$ on ${}^X\!/\!{}_{\sim_f}$ by $O \in {}^\tau\!/\!{}_{\sim_f} \Leftrightarrow \cup_{[x] \in O}[x] \in \tau$. We will check in a following proposition that ${}^\tau\!/\!{}_{\sim_f}$ is in fact a topology on ${}^X\!/\!{}_{\sim_f}$. With this construction, we then have that $({}^X\!/\!{}_{\sim_f}, {}^\tau\!/\!{}_{\sim_f})$ is homeomorphic to (Y, τ_f) so we can (and do) view (Y, τ_f) as the quotient space $({}^X\!/\!{}_{\sim_f}, {}^\tau\!/\!{}_{\sim_f})$. Of course, we have several things to now check.

Exercise 4.4.3. Let $f : X \to Y$ and define a relation $\sim_f$ on X by $x_1 \sim_f x_2 \Leftrightarrow f(x_1) = f(x_2)$. Prove $\sim_f$ is an equivalence relation on X.

Proposition 4.4.4. *Let (X, τ) be a topological space, Y a set, and $f : (X, \tau) \to Y$ be onto. Define ${}^X\!/\!{}_{\sim_f} = \{[x] | \, x \in X\}$, where $\sim_f$ is the equivalence relation given in Exercise 4.4.3. Define ${}^\tau\!/\!{}_{\sim_f}$ to be the collection of subsets O of ${}^X\!/\!{}_{\sim_f}$ where*

$$O \in {}^\tau\!/\!{}_{\sim_f} \Leftrightarrow \bigcup_{[x] \in O} [x] \in \tau.$$

Then ${}^\tau\!/\!{}_{\sim_f}$ is a topology on ${}^X\!/\!{}_{\sim_f}$.

Proof. First, $\emptyset \in {}^{\tau}/_{\sim_f}$ since $\cup_{[x]\in\emptyset}[x] = \emptyset \in \tau$. Also, ${}^{X}/_{\sim_f} \in {}^{\tau}/_{\sim_f}$ since $\cup_{[x]\in X}[x] = X \in \tau$.

Next, let $O_i \in {}^{\tau}/_{\sim_f}$ for all $i \in I$. Then $\cup_{[x]\in O_i}[x] \in \tau$, for all $i \in I$. Then

$$\bigcup_{[x]\in\cup_{i\in I}O_i} [x] = \bigcup_{i\in I}\left(\bigcup_{[x]\in O_i}[x]\right) \in \tau$$

since it is an arbitrary union of elements of τ. Hence, $\cup_{i\in I}O_i \in {}^{\tau}/_{\sim_f}$.

Finally, let $n \in \mathbb{Z}_+$ and $O_1, O_2, \ldots, O_n \in {}^{\tau}/_{\sim_f}$. Then $\cup_{[x]\in O_i}[x] \in \tau$, for all $i = 1, 2, \ldots, n$ and so

$$\bigcup_{[x]\in\cap_{i=1}^{n}O_i} [x] = \bigcap_{i=1}^{n}\left(\bigcup_{[x]\in O_i}[x]\right) \in \tau$$

since it is a finite intersection of elements from τ. Hence, $\cap_{i=1}^{n}O_i \in {}^{\tau}/_{\sim_f}$.

Thus, ${}^{\tau}/_{\sim_f}$ is a topology on ${}^{X}/_{\sim_f}$. □

Lastly, we have to check that we do, in fact, get the homeomorphism mentioned earlier between $({}^{X}/_{\sim_f}, {}^{\tau}/_{\sim_f})$ and (Y, τ_f).

Theorem 4.4.5. *Let (X, τ) be a topological space, Y a set, and $f : (X, \tau) \to Y$ be onto. Then $({}^{X}/_{\sim_f}, {}^{\tau}/_{\sim_f})$ is homeomorphic to (Y, τ_f).*

Proof. Define $h : (Y, \tau_f) \to ({}^{X}/_{\sim_f}, {}^{\tau}/_{\sim_f})$ by $h(y) = f^{-1}(y)$, for all $y \in Y$. Since f is onto, h is well-defined. Clearly, h is a bijection, so it suffices to show that h and h^{-1} are continuous. Let's first show h is continuous. To this end, let $O \in {}^{\tau}/_{\sim_f}$. We want to show $h^{-1}(O) \in \tau_f$. By the definition of τ_f, it suffices to show $f^{-1}(h^{-1}(O)) \in \tau$.

Recall by the definition of $\tau/\sim_f$ that we have $\cup_{[x]\in O}[x] \in \tau$. Then,

$$\begin{aligned} h^{-1}(O) &= \{y \in Y | f^{-1}(y) \in O\} \\ &= \{f(x_0) |\, [x_0] \in O\} \\ &= \left\{ f(x_0) \,\middle|\, x_0 \in \bigcup_{[x]\in O} [x] \right\} \\ &= f\left(\bigcup_{[x]\in O} [x] \right) \end{aligned}$$

and so, $f^{-1}(h^{-1}(O)) = f^{-1}(f(\cup_{[x]\in O}[x])) = \cup_{[x]\in O}[x] \in \tau$, and so $h^{-1}(O) \in \tau_f$.

Now, to show h^{-1} is continuous, let $O \in \tau_f$. Then we know, by the definition of τ_f, that $f^{-1}(O) \in \tau$. We want to show that $h(O) \in \tau/\sim_f$, i.e., that $\cup_{[x]\in h(O)}[x] \in \tau$. Note that $h(O) = \{f^{-1}(y) |\, y \in O\}$, so

$$\bigcup_{[x]\in h(O)} [x] = \bigcup_{y \in O} f^{-1}(y) = f^{-1}(O) \in \tau.$$

Therefore, $h(O) \in \tau/\sim_f$ and hence h^{-1} is continuous. Therefore, h is a homeomorphism between (Y, τ_f) and $(X/\sim_f, \tau/\sim_f)$. □

Suppose we have $f : (X, \tau) \to (Y, \sigma)$ which is onto and continuous. Proposition 4.4.2 then tells us that $\sigma \subseteq \tau_f$. The next proposition gives conditions for $\sigma = \tau_f$ but we first need a couple of definitions.

Definition 4.4.6. Let $f : (X, \tau) \to (Y, \sigma)$, where (X, τ) and (Y, σ) are topological spaces. If $f(O) \in \sigma$ for all $O \in \tau$, then we say the function f is **open**. If $f(C)$ is closed for all closed sets C, then we say the function f is **closed**.

Thus, a function is open if it maps open sets to open sets and is closed if it maps closed sets to closed sets. We are now ready for the above mentioned proposition.

Proposition 4.4.7. *Let (X, τ) and (Y, σ) be topological spaces and suppose $f : (X, \tau) \to (Y, \sigma)$ is continuous. If f is open or closed, then $\sigma = \tau_f$, where τ_f is the quotient topology on Y induced by f.*

Proof. We already know, by Proposition 4.4.2, that $\sigma \subseteq \tau_f$.First, suppose f is open. Let $O \in \tau_f$. By construction of τ_f, we have that $f^{-1}(O) \in \tau$ and, since f is open, $O = f(f^{-1}(O) \in \sigma$ and so $\sigma = \tau_f$. Now, suppose f is closed. Let $O \in \tau_f$. By construction of τ_f, we have that $f^{-1}(O) \in \tau$. Hence, $f^{-1}(O)^c$ is closed with respect to τ. Since f is closed, $f(f^{-1}(O)^c)$ is closed with respect to σ and, since $O^c = f(f^{-1}(O)^c)$, we have that O^c is closed with respect to σ. That is, $O \in \sigma$. □

Note that when we combine Proposition 4.4.7 with Theorem 4.4.5, we get the following corollary.

Corollary 4.4.8. *Let (X, τ) and (Y, σ) be topological spaces. If $f : (X, \tau) \to (Y, \sigma)$ is onto, continuous, and either open or closed, then (Y, σ) is homeomorphic to $({}^{X}/_{\sim_f}, {}^{\tau}/_{\sim_f})$.*

Proof. The proof follows immediately from Proposition 4.4.7 and Theorem 4.4.5 □

Admittedly, this section has been rather technical. The diligent reader will hopefully find the next list of examples well worth the hard work.

Example 4.4.9. (i) Consider $X = [0, 2\pi]$ with the usual topology τ (technically, $\tau = \eta|_{[0,2\pi]}$, where η is the usual topology on $\mathbb{R}$) and $Y = S_1$ with the usual topology σ. Define $f : (X, \tau) \to (Y, \sigma)$ by $f(t) = (\cos(t), \sin(t))$. Then f is onto, continuous, and open. Hence, by Corollary 4.4.8, $({}^{[0,2\pi]}/_{\sim_f}, {}^{\tau}/_{\sim_f})$ is homeomorphic to (S_1, σ). Now, if we think about the elements of ${}^{[0,2\pi]}/_{\sim_f}$, we will see that

$$ {}^{[0,2\pi]}/_{\sim_f} = \{\{t\} | \, 0 < t < 2\pi\} \cup \{\{0, 2\pi\}\}. $$

So, ${}^{[0,2\pi]}/_{\sim_f}$ is essentially $[0, 2\pi]$ except that we make no distinction between 0 and 2π. Further, the topology ${}^{\tau}/_{\sim_f}$ is essentially τ except that it also makes no distinction between 0 and 2π. Hence, we get that S_1 is homeomorphic to $[0, 2\pi]$ as long as we "identify" the endpoints 0 and 2π.

(ii) First note that $S_1 \times [0, 2\pi]$ is a cylinder in $\mathbb{R}^3$ with radius 1 and height 2π. Equip the cylinder with the usual topology σ. Now, define $f : [0, 2\pi] \times [0, 2\pi] \to S_1 \times [0, 2\pi]$ by $f(t, s) = (\cos(t), \sin(t), s)$ where we equip $[0, 2\pi] \times [0, 2\pi]$ with the usual topology τ. Then f is continuous, onto, and open so $S_1 \times [0, 2\pi]$ is a quotient space of $[0, 2\pi] \times [0, 2\pi]$ and

$$\left({}^{[0,2\pi]\times[0,2\pi]}/_{\sim_f}, {}^{\tau}/_{\sim_f}\right) \cong \left(S_1 \times [0, 2\pi], \sigma\right).$$

Further, the equivalence classes of the quotient space are given by

$${}^{[0,2\pi]\times[0,2\pi]}/_{\sim_f} = \{\{(t, s)\} \mid 0 < t < 2\pi\,,\ s \in [0, 2\pi]\} \cup \{\{(0, s), (2\pi, s)\} \mid s \in [0, 2\pi]\}\,.$$

Hence, the elements of ${}^{[0,2\pi]\times[0,2\pi]}/_{\sim_f}$ are precisely the elements of $[0, 2\pi] \times [0, 2\pi]$ except we identify points of the form $(0, s)$ to points of the form $(2\pi, s)$, for all $s \in [0, 2\pi]$. Graphically, we picture $[0, 2\pi] \times [0, 2\pi]$ as a square. If we then make the identifications mentioned, it is identifying the left edge of the square to the right edge, thus creating a cylinder.

(iii) Let us now avoid the technical aspects of these constructions and focus on identifying different edges of the square $[0, 2\pi] \times [0, 2\pi]$ with each other. First, identify the left edge of the square with the right edge, as we did in the above example, to form a cylinder. Then identify the bottom edge with the top edge so that points of the form $(t, 0)$ are identified with points of the form $(t, 2\pi)$. We then bend the cylinder around to form a donut shape, called the **torus**. The torus is often denoted by $\mathbb{T}$ or $S_1 \times S_1$ since the quotient map for this construction is $f : [0, 2\pi] \times [0, 2\pi] \to S_1 \times S_1$, where $f(t, s) = (\cos(t), \sin(t), \cos(s), \sin(s))$.

(iv) Suppose instead we start with the square $[0, 2\pi] \times [0, 2\pi]$ and identify the bottom edge with the top edge but we twist the bottom edge before doing so. Technically, we are identifying points of the form $(t, 0)$ with points of the form $(2\pi - t, 2\pi)$. We then end up with a topological object called the **Möbius strip**.

(v) Now, lets start with the square $[0, 2\pi] \times [0, 2\pi]$, identify the left edge with the right edge to form a cylinder. We will then identify the top edge with the bottom edge but twist the bottom edge before doing so. This then creates an object called the **Klein bottle**. It is easiest to picture by imagining, once we have formed the cylinder, picking up the bottom from it's circular edge, stretching it out, inserting it through the side of the cylinder, up through the top of the cylinder, and then matching it to the top circle by curling the sides down. Unfortunately, this interpretation of the Klein bottle is misleading as it does not actually intersect itself like this. Just like the torus, the image of the quotient map is actually a subset of $\mathbb{R}^4$ so the Klein bottle is avoiding this self-intersection by using a fourth dimension.

(vi) Lastly, as you can probably guess, take the square $[0, 2\pi] \times [0, 2\pi]$ once more, pick up the left edge and twist it, before identifying it with the right edge. Then, pick up the bottom edge and twist it, before identifying it with the top edge. The resulting object is called the **projective plane** (or real projective plane) and is denoted most often by $\mathbb{R}P^2$. This is a rather difficult object to try to picture. Part of the issue is that, like the Klein bottle, it cannot be properly thought of as a subset of $\mathbb{R}^3$.

Exercise 4.4.10. Give quotient maps for the Möbius strip, Klein bottle, and projective plane.

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