MATH 422-Introduction to Topology

Matthew Ziemke

January 27, 2026

Contents

Chapter 1

Preliminaries

Chapter 2

Bridging the Gap: Metric Spaces to Topological Spaces

Chapter 3

Introduction to Topological Spaces

In the last chapter we saw that one very important reason to concern ourselves with topological spaces is to introduce an analytic structure to more general sets of objects or to change the analytic structure on sets which already have one. With this framework, one could say that to study topology is to study analysis in its most abstract setting. Certainly, many mathematicians take this point of view, especially those who primarily work in analysis or analysis adjacent fields. It should be stated though that another perspective is that topology is an abstraction of geometry. Of course, even with our very limited understanding of topology up to this point, we can see some overlap as we have already talked about distance between two points, open disks, open squares, the Cartesian plane, etc. Nevertheless, the entirety of this perspective is probably not clear to the reader but we will attempt to address it in more detail as we progress through these notes.

To continue with our discussion from the last chapter, we found a way to address analytic properties without having to mention a metric but rather open sets. We decided that if simply designate which subsets are open, then we can define certain analytic properties to sequences and functions. If we look back at the definition of a metric, we didn't want any function $d : X \times X \to \mathbb{R}$ to be a metric on X. We wanted d to satisfy certain properties we would expect distance to satisfy. The same is true in defining our open sets (which, in so doing, defines a topology). We want the open sets to satisfy certain properties that we would expect open sets to satisfy. Admittedly, these properties are not as obvious as the defining properties of metrics but, after much debate and trial and error, these are the properties for which we have decided to be the most appropriate.

3.1 Definitions and Examples

Definition 3.1.1. Let X be a nonempty set. We say a set $\tau \subseteq \mathcal{P}(X)$ is a **topology** on X if it satisfies the following properties:

(i) $\emptyset, X \in \tau$

(ii) τ is closed under the formation of arbitrary unions. That is, if I is an index set and $A_i \in \tau$, for all $i \in I$, then $\cup_{i \in I} A_i \in \tau$, and

(iii) τ is closed under the formation of finite intersections. That is, if $n \in \mathbb{Z}_+$, and $A_1, A_2, \ldots, A_n \in \tau$, then $\cap_{i=1}^{n} A_i \in \tau$.

The elements of τ are called **open sets**. The ordered pair (X, τ) is called a **topological space**. If $x \in X$ and $U \in \tau$ such that $x \in U$ then we call U an **open neighborhood of** x.

As I mentioned, it is not completely clear at this point, why we require our open sets to have these defining properties. It is worth recalling though that in a metric space, our open sets satisfy these properties. So, at the very least, these properties should not be surprising.

Example 3.1.2. (i) Let X be a nonempty set and let $\tau = \{\emptyset, X\}$. Then τ is a topology on X called the **trivial topology** or **indiscrete topology**. Note here that the only open sets are $\emptyset$ and X.

(ii) Let X be a nonempty set and let $\tau = \mathcal{P}(X)$. Then τ is a topology on X called the **discrete topology**. Every set here is open. When we say a field of mathematics is "discrete" or, when we study Discrete Mathematics, we're studying mathematical objects without any topological structure. Imposing the discrete topology on a set X essentially gives it no topological structure since defining every set to be open essentially renders the topological structure useless.

(iii) Let (X, d) be a metric space and let τ be the set of all open sets as they were defined in Chapter 2. Then, from a theorem from Chapter 2, τ is a topology on X called the **metric topology on** X **induced by** d. If (X, τ) is a topological space and their exists a metric d such that the topology induced by d equals τ, then we say the topological space (X, τ) is **metrizable**.

(iv) Consider $\mathbb{R}$ with the usual metric d and let τ be the metric induced by d. Then τ is called the **usual topology on** $\mathbb{R}$. The elements of τ are precisely the open sets we discuss in high school and our calculus classes. For example, the intervals of the form (a, b) are open as is any union of intervals of this form. Intervals of the form $(a, b]$, $[a, b)$, and $[a, b]$ are not open.

(v) Consider the set $\mathbb{R}$ and define $\tau = \{(a, \infty) : a \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$. Then τ is a topology on $\mathbb{R}$ and it's called the **right ray topology**. It is an exercise below to prove τ is a topology.

(vi) Let X be a nonempty set and define τ to be the collection of all subsets of X whose complement is finite. That is, $A \in \tau \Leftrightarrow A^c$ is finite. Then τ is a topology on X called the **finite complement topology**.It is an exercise below to prove τ is a topology. Note that if X is a finite set then τ equals the discrete topology on X.

(vii) Let X be a nonempty set and fix an $x_0 \in X$. Define $\tau_{x_0} = \{A \in \mathcal{P}(X) : x_0 \in A\} \cup \{\emptyset\}$. Then τ_{x_0} is a topology on X called the **distinguished point topology**. It is a following exercise to prove τ_{x_0} is a topology.

Exercise 3.1.3. (i) Prove the right ray topology is a topology on $\mathbb{R}$.

(ii) Let X be an infinite set. Prove the finite complement topology is a topology on X.

(iii) Let X be a nonempty set and let $x_0 \in X$. Prove that the distinguished topology, τ_{x_0}, is a topology on X.

It is often easier to define a topology by defining the sets which make up all of the open sets. For example, in a metric space (X, d), we saw that every open set can be written as the union of sets of the form $B_d(x,r)$ for some $x \in X$ and $r > 0$. We also used the terminology that the collection of all sets of the form $B_d(x,r)$ is a "base for the topology on X" and we called the sets of the form $B_d(x,r)$ "basic open sets." We carry these concepts and terminology to general topological spaces.

Definition 3.1.4. Let (X,τ) be a topological space and let $\mathcal{B} \subseteq \tau$. We say $\mathcal{B}$ forms a **base for the topology** if every element of τ can be written as a union of elements of $\mathcal{B}$. We then call the elements of $\mathcal{B}$ **basic open sets**. If $x \in X$ and $U \in \mathcal{B}$ such that $x \in U$, then we call U a **basic open neighborhood of** x.

Example 3.1.5. (i) As discussed, if (X,d) is a metric space and $\mathcal{B}$ is the collection of all sets of the form $B_d(x,r)$, where $x \in X$ and $r > 0$, then $\mathcal{B}$ forms a base for (X,τ) where τ is the metric topology on X induced by d.

(ii) Consider $\mathbb{R}$ with the usual topology and let $\mathcal{B} = \{(a,b) : a,b \in \mathbb{R} \text{ and } a < b\}$. Then $\mathcal{B}$ forms a base for $\mathbb{R}$. Note that an empty union of sets equals the empty set.

Proposition 3.1.6. *Let (X,τ) be a topological space and suppose $\mathcal{B}$ is a base for τ. Let O be an open neighborhood of x. Then there exists $U \in \mathcal{B}$ such that $x \in U \subseteq O$.*

Proof. Let $x \in X$ and let O be an open neighborhood of x. Since $O \in \tau$ and $\mathcal{B}$ is a base for τ, there exists an index set I and $B_i \in \mathcal{B}$ such that $O = \cup_{i \in I} B_i$. Since $x \in O$, we have that $x \in B_{i_0}$ for some $i_0 \in I$. Let $U = B_{i_0}$. Then $x \in U \subseteq O$. □

Now, suppose we have a set X (without a topology) and we would like to define a topology by defining the basic open sets. One might be tempted to think that it is enough to define the sets in $\mathcal{B}_0$ and then define the elements of τ_0 to be the collection of all unions of elements from $\mathcal{B}_0$. But this will not always make τ_0 a topology. We need the elements of $\mathcal{B}_0$ to satisfy two properties as the next theorem shows.

Theorem 3.1.7. *Let X be a nonempty set (without a topology) and let $\mathcal{B}$ be a family of subsets of X satisfying the properties:*

(i) For every $x \in X$ there exists $U \in \mathcal{B}$ such that $x \in U$, and

(ii) For every $U_1, U_2 \in \mathcal{B}$ and every $x \in U_1 \cap U_2$ there exists $U_3 \in \mathcal{B}$ such that $x \in U_3 \subseteq U_1 \cap U_2$.

Now, let τ be the collection of subsets of X consisting of all unions of elements of $\mathcal{B}$. Then τ is a topology on X and $\mathcal{B}$ is a base for τ.

Proof. First, $\emptyset \in \tau$ since we can write it as an empty union. Further, by property (i), for all $x \in X$, there exists $U_x \in \mathcal{B}$ such that $x \in U_x$. Then, $X = \cup_{x \in X} U_x \in \tau$.

Now, let I be an index set and suppose $O_i \in \tau$, for all $i \in I$. For each $i \in I$, there exists an index set K_i such that $O_i = \cup_{k \in K_i} B_{k,i}$. Then,

$$\bigcup_{i \in I} O_i = \bigcup_{i \in I} \bigcup_{k \in K_i} B_{k,i} \in \tau,$$

since it is an arbitrary union of elements from $\mathcal{B}$. Hence, τ is closed under arbitrary unions.

Next, we will show that the intersection of two elements of τ is an element of τ. To this end, let $O_1, O_2 \in \tau$. Let $x \in O_1 \cap O_2$. Since $x \in O_1$, which can be written as a union of elements from $\mathcal{B}$, there exists $B_1 \in \mathcal{B}$ such that $x \in B \subseteq O_1$. Similarly, there exists $B_2 \in \mathcal{B}$ such that $x \in B_2 \subseteq O_2$. By property (ii), we then have that there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. Thus, for every $x \in O_1 \cap O_2$, we can find $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq O_1 \cap O_2$. Thus,

$$O_1 \cap O_2 = \cup_{x \in O_1 \cap O_2} B_x \in \tau.$$

We will now use induction to show that any finite intersection of elements from τ is an element of τ. Let $n \in \mathbb{Z}_+$. The case when $n = 1$ is trivial and we proved the $n = 2$ case above. Suppose that for some $k \in Z_+$, we have that if $O_1, O_2, \ldots, O_k \in \tau$, then $\cap_{i=1}^{k} O_i \in \tau$. Let $O_1, O_2, \ldots, O_{k+1} \in \tau$. By hypothesis, we have that $\cap_{i=1}^{k} O_i \in \tau$ and, since $O_{k+1} \in \tau$, by the $n = 2$ case, we have that

$$\cap_{i=1}^{k+1} O_i = \left(\cap_{i=1}^{k} O_i\right) \cap O_{k+1} \in \tau.$$

Thus, τ is closed under finite intersections. $\square$

Definition 3.1.8. If X is a set and $\mathcal{B}$ is a collection of subsets of X satisfying properties (i) and (ii) above then the topology τ constructed above is called the **topology on X generated by the base $\mathcal{B}$**.

Note that if (X, τ) is a topological space and β is a base for τ, then β satisfies properties (i) and (ii) of Theorem 3.1.7. It follows directly from Proposition 3.1.6.

Example 3.1.9. (i) Consider the set $\mathbb{R}$ and let $\mathcal{B}$ be the set of all intervals of the form $[a, b)$, where $a, b \in \mathbb{R}$ and $a < b$. Note that $\mathcal{B}$ is not itself a topology on $\mathbb{R}$ as, for example, $[1, 2), [3, 4) \in \mathcal{B}$ but $[1, 2) \cup [3, 4) \notin \mathcal{B}$. However, we can easily see that $\mathcal{B}$ satisfies properties (i) and (ii) of Theorem 3.1.7. Indeed, for any $x \in \mathbb{R}$, $x \in [x, x + 1) \in \mathcal{B}$ so property (i) is satisfied. Given $[a, b), [c, d) \in \mathcal{B}$, if $[a, b) \cap [c, d) = \emptyset$ then there is nothing to prove. Suppose $[a, b) \cap [c, d) \neq \emptyset$ and let $x \in [a, b) \cap [c, d)$. Let $m = \min\{b, d\}$. Then $[x, m) \subseteq [a, b)$ and $[x, m) \subseteq [c, d)$, thus $[x, m) \in \mathcal{B}$ and $x \in [x, m) \subseteq [a, b) \cap [c, d)$. Thus, if we define τ to be the collection of all subsets of $\mathbb{R}$ consisting of unions of sets from $\mathcal{B}$, then τ is a topology on $\mathbb{R}$ and $\mathcal{B}$ is a base for τ. The topology τ is called the **left-hand topology on $\mathbb{R}$**.

(ii) Let X with the relation $\leq$ be a totally ordered set with at least two elements. Define the intervals $(a, \infty) = \{x \in X |\, a \leq x \text{ and } a \neq x\}$, $(-\infty, a) = \{x \in X |\, x \leq a \text{ and } x \neq a\}$, and $(a, b) = \{x \in X |\, a \leq x\,,\ x \leq b\,,\ x \neq a\,,\ \text{and } x \neq b\}$ for all $a, b \in X$. Then these intervals, for all $a, b \in X$, form a base for a topology τ on X called the **order topology**.

Exercise 3.1.10. Consider the relation R on $\mathbb{R}^2$ defined by $(a,b)R(c,d)$ if $a < c$ or, if $a = c$, then $b \leq d$. Prove that $\mathbb{R}^2$ with the relation R is a totally ordered set (this order on $\mathbb{R}^2$ is called the **lexicographic order**). If we then equip $\mathbb{R}^2$ with the order topology, basic open sets would be "intervals" of the form $((a,b),\infty)$, $(-\infty,(a,b))$, and $((a,b),(c,d))$, for all $(a,b),(c,d) \in \mathbb{R}^2$. Draw pictures of the open intervals $((1,2),(7,-5))$ and $((1,2),(1,8))$.

Definition 3.1.11. Let X be a set and let τ_1 and τ_2 be topologies on X. If $\tau_1 \subseteq \tau_2$, we say τ_1 is **weaker** or **coarser** than τ_2 and we say τ_2 is **stronger** or **finer** than τ_1.

Example 3.1.12. If τ_1 is the trivial topology on $\mathbb{R}$ and τ_2 is the usual topology on $\mathbb{R}$, then $\tau_1 \subseteq \tau_2$ so τ_1 is weaker than τ_2 or τ_2 is stronger than τ_1. If τ_3 is the distinguished point topology on $\mathbb{R}$, for some $x_0 \in \mathbb{R}$, then $\tau_2 \nsubseteq \tau_3$ and $\tau_3 \nsubseteq \tau_2$ so neither topology is stronger or weaker than the other. Another way to say this is that τ_2 and τ_3 are incomparable.

While a base for a topology tells us what each element of the topology looks like, we can take this idea a step further and discuss a subbase, which shows us what each basic open set looks like.

Definition 3.1.13. Let (X,τ) be a topological space. We say a subset $\mathcal{C}$ of τ forms a **subbase** for the topology τ if

$$\mathcal{B} = \{\cap_{i=1}^{n} C_i | n \in \mathbb{Z}_+ \text{ and } C_i \in \mathcal{C}\}$$

forms a base for τ.

Note that if we want a collection of subsets of a set X to form a base for a topology on X, then we have to satisfy the properties (i) and (ii) as discussed in Theorem 3.1.7. We do not have to do anything like this for a subbase. Given any collection of subsets of a set X, it will form a subbase for a topology on X, as the next proposition shows.

Proposition 3.1.14. *Let X be a set and let $\mathcal{C} \subseteq \mathcal{P}(X)$. Then $\mathcal{C}$ is a subbase for a topology τ on X. Further, τ is the weakest topology containing $\mathcal{C}$ (by this we mean, if τ_1 is a topology on X and $\mathcal{C} \subseteq \tau_1$ then $\tau \subseteq \tau_1$).*

Proof. Let X be a set and let $\mathcal{C}$ be a family of subsets of X. Let

$$\mathcal{B} = \{\cap_{i=1}^{n} C_i | n \in \mathbb{Z}_+ \text{ and } C_i \in \mathcal{C}\}.$$

We have to show $\mathcal{B}$ satisfies properties (i) and (ii) of Theorem 3.1.7. First, $\mathcal{B}$ includes empty intersections. Since an empty intersection equals X, we have that $X \in \mathcal{B}$. Hence, for any $x \in X$, there exists an element of $\mathcal{B}$ which contains x (namely, X). Hence, $\mathcal{B}$ satisfies property (i).

For property (ii), let $U_1, U_2 \in \mathcal{B}$ and $x \in U_1 \cap U_2$. Since $U_1, U_2 \in \mathcal{B}$, there exist $n, m \in \mathbb{Z}_+$ and $C_i, D_j \in \mathcal{C}$, for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, such that $U_1 = \cap_{i=1}^{n} C_i$ and $U_2 = \cap_{j=1}^{m} D_j$. So, let $U_3 = (\cap_{i=1}^{n} C_i) \cap \left(\cap_{j=1}^{m} D_j\right) \in \mathcal{B}$. Then, $x \in U_3 \subseteq U_1 \cap U_2$ and so property (ii) is satisfied as well. □

Definition 3.1.15. Let X is a set and $\mathcal{C} \subseteq \mathcal{P}(X)$. Then, as we saw above, $\mathcal{C}$ is a subbase for a topology τ on X. We call τ the **topology generated by the subbase** $\mathcal{C}$ and it is the weakest topology on X containing $\mathcal{C}$.

As the reader will see later in the notes, using subbases is a convenient way to define a topology. We simply have to specify which sets are in our subbase and then we can generate the weakest topology which contains those sets. A word of caution though, while it is often nice to be able to pick which sets we want to be open and then generate a topology which includes those sets, it does generate the *weakest* topology which does so. This can leave us with a topology which isn't particularly robust. For example, if, for whatever reason, we would like a topology on $\mathbb{R}$ which makes the interval $[1,2]$ an open set, and we let $\mathcal{C} = \{[1,2]\}$, then the base we will get is $\beta = \{\mathbb{R}, [1,2]\}$, and the topology we will end up with is $\tau = \{\emptyset, \mathbb{R}, [1,2]\}$ which is not particularly interesting.

3.2 Closed Sets

Definition 3.2.1. Let (X, τ) be a topological space. We say a set $A \subseteq X$ is **closed** if $A^c \in \tau$.

Example 3.2.2. (i) Consider $\mathbb{R}$ with the usual topology. Any set of the form $[a, b]$, where $a < b$, is closed since $[a, b]^c = (-\infty, a) \cup (b, \infty)$ which is open since it is the union of two open sets. Any set of the form $(a, b]$, where $a < b$ is not open, as we have seen. It is also not closed since $(a, b]^c = (-\infty, a] \cup (b, \infty)$ which is not open. Further, we know $\emptyset$ and $\mathbb{R}$ are both open. Hence $\emptyset$ and $\mathbb{R}$ are both closed, since $\emptyset^c = \mathbb{R}$ and $\mathbb{R}^c = \emptyset$. Hence, it is possible for a subset of $\mathbb{R}$ to be open, closed, both open and closed, or neither open nor closed.

(ii) Consider any nonempty set X with the discrete topology. Since every set is open, the complement of every set is open, and so every set is closed.

(iii) Let X be a nonempty set and $x_0 \in X$. Equip X with the distinguished point topology τ_{x_0}. The open sets are $\emptyset$ and any set containing x_0 so the closed sets are X and any set which does not contain x_0. Note here that besides $\emptyset$ and X (which are both open and closed), a set is either open or closed, depending on whether or not it contains x_0.

Definition 3.2.3. Let (X, τ) be a topological space and let $A \subseteq X$. If $A \in \tau$ and $A^c \in \tau$, then we say A is **clopen**.

Exercise 3.2.4. Let (X, τ) be a topological space.

(i) Let I be an index set and let A_i be a closed set, for all $i \in I$. Prove $\cap_{i \in I} A_i$ is closed.

(ii) Let $n \in \mathbb{Z}_+$ and let $A_1, A_2, \ldots, A_n$ be closed sets. Prove $\cup_{i=1}^n A_i$ is closed.

(iii) Give an example of closed sets $A_1, A_2, \ldots$ such that $\cup_{i=1}^\infty A_i$ is not closed.

Definition 3.2.5. Let (X, τ) be a topological space and let $A \subseteq X$. The **closure** of A in X is the set

$$\overline{A} = \bigcap \{C \subseteq X | C \text{ is closed and } A \subseteq C\}.$$

The **interior** of A in X is the set

$$A^o = \bigcup \{U \subseteq X : U \text{ is open and } U \subseteq A\}.$$

Theorem 3.2.6. *Let (X, τ) be a topological space and let A and B be subsets of X. Then*

(i) If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$,

(ii) $\overline{\overline{A}} = \overline{A}$,

(iii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$,

(iv) $\overline{\emptyset} = \emptyset$, and

(v) A is closed if and only if $A = \overline{A}$.

Proof. First note that for any $C \subseteq X$, we have that $\overline{C}$ is closed since it is defined as an intersection of closed sets.

(i) Let $A \subseteq B$. Since $A \subseteq B \subseteq \overline{B}$, we have that $\overline{B}$ is a closed set containing A. Since $\overline{A}$ is defined to be the intersection of all closed sets which contain A, we have that $\overline{A} \subseteq \overline{B}$.

(iii) Well, $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$ so $A \cup B \subseteq \overline{A} \cup \overline{B}$. Hence, $\overline{A} \cup \overline{B}$ is a closed set containing $A \cup B$, so $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. For the other inclusion, we have that $A \subseteq A \cup B$ so, by (i), we have $\overline{A} \subseteq \overline{A \cup B}$. Similarly, $B \subseteq A \cup B$ so $\overline{B} \subseteq \overline{A \cup B}$. Hence $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

(v) First, suppose A is closed. Since every set is contained in its closure, we have that $A \subseteq \overline{A}$. Since $\overline{A}$ is the intersection of all closed sets which contain A and A is a closed set which contains itself, we have that $\overline{A} \subseteq A$ and so $\overline{A} = A$. Now suppose $\overline{A} = A$. Since $\overline{A}$ is defined as an intersection of closed sets, we have that $\overline{A}$ is closed. Since $A = \overline{A}$, we then have that A is closed.

(ii) For any set A, we know $\overline{A}$ is closed. Then, by (v), we have that $\overline{\overline{A}} = \overline{A}$.

(iv) The empty set is closed, so by (v) we have $\overline{\emptyset} = \emptyset$. □

Exercise 3.2.7. Let (X, τ) be a topological space and let A and B be subsets of X. Prove the following statements.

(i) If $A \subseteq B$, then $A^o \subseteq B^o$.

(ii) $A^0 \subseteq A$.

(iii) $(A^o)^o = A^o$.

(iv) $(A \cap B)^o = A^0 \cap B^0$.

(v) $X^o = X$.

(vi) A is open if and only if $A = A^o$.

Exercise 3.2.8. Let (X, d) be a metric space. Define the **closed ball** centered at x or radius $r > 0$ to be $\overline{B}_d(x, r) = \{y \in X : d(x, y) \leq r\}$. Prove that for all $x \in X$ and $r > 0$, $\overline{B}_d(x, r)$ is closed and $\overline{B_d(x, r)} = \overline{B}_d(x, r)$.

Another way to discuss closed sets and the closure of sets is to introduce the concept of limit points. We will start with the definition.

Definition 3.2.9. Let (X, τ) be a topological space and let $A \subseteq X$. We say $x \in X$ is a **limit point** of A if every open neighborhood of x intersects A at a point other than x. Said differently, x is a limit point for A if, for every open neighborhood U of x, we have that $(U \cap A) \setminus \{x\} \neq \emptyset$.

In the definition above, note that elements of A are not necessarily limit points of A. We want to avoid isolated elements of A being considered limit points. For example, if $A = [0, 1) \cup \{2\}$, the element 2, while in A, is not a limit point of A. Note also, that while 1 is not in A, it is a limit point of A.

The next theorem gives us a different way to characterize closed sets and the closure of sets. It is sometimes given as the definition of a closed set, particularly in analysis textbooks. For us, it is a theorem.

Theorem 3.2.10. *Let (X, τ) be a topological space and let $A \subseteq X$. Let A' be the set of all limit points of A. Then $\overline{A} = A \cup A'$.*

Proof. Let $x \in A'$ and suppose $x \notin \overline{A}$. Then there exists a closed set C such that $A \subseteq C$ and $x \notin C$. Hence, $x \in C^c$ and C^c is open, so C^c is an open neighborhood of x. However, $\overline{A} \subseteq C$ so $C^c \cap \overline{A} = \emptyset$ which contradicts the fact that x is a limit point of A. Thus, we must have that $x \in \overline{A}$ and so $A' \subseteq \overline{A}$. Since $A \subseteq \overline{A}$, we then have that $A \cup A' \subseteq \overline{A}$.

Now, suppose $x \in \overline{A}$. If $x \in A$ then we are done so suppose $x \notin A$. We want to show $x \in A'$. Suppose not. Then there exists an open neighborhood U of x such that $A \cap U = \emptyset$, since $x \notin A$. So, U^c is a closed set and $A \subseteq U^c$. Thus, $\overline{A} \subseteq U^c$ but $x \notin U^c$. This contradicts the fact that $x \in \overline{A}$. Hence, we must have that $x \in A'$ and so $\overline{A} \subseteq A \cup A'$. □

One thing the above theorem tells us is that to take the closure of a set, we simply have to include all of its limit points. This helps us to understand the difference between a set and its closure along with the elements gained by taking the closure. Another thing the above theorem tells us is a different way to understand closed sets. Notice that the theorem implies a set is closed if and only if it contains all of its limit points.

3.3 Sequences in Topological Spaces

Definition 3.3.1. Let (X, τ) be a topological space and let $(x_n)_{n=1}^{\infty}$ be a sequence in X. For some $x \in X$, we say the sequence $(x_n)_{n=1}^{\infty}$ converges to x, and write $x_n \xrightarrow{\tau} x$, if for every open neighborhood O of x there exists $N \in \mathbb{Z}_+$ such that, if $n \geq N$, we have $x_n \in O$.

From our discussion in the previous chapter, this definition agrees with the definition of convergent sequences in metric spaces when (X, d) is a metric space and τ is the topology induced by the metric d. Because of this, it also agrees with our definition of convergent sequences from calculus class when $X = \mathbb{R}$ and τ is the usual metric on $\mathbb{R}$.

If we consider $\mathbb{R}$ with the usual topology and wish to show, for example, that the sequence $x_n = \frac{1}{n}$ converges to 0 using the definition above, we want to show that for *any* open neighborhood O of 0, there exists an $N \in \mathbb{Z}_+$ such that for all $n \geq N$, we have $x_n \in O$. This would not only require us to check open sets of the form (a, b) containing 0 but also open sets like, for example, $(-1, 1) \cup (3, 14) \cup (28, 112)$, which is frustrating, but thankfully, unnecessary as the next theorem tells us that if the topology has a base, then we do not have to check all of the open neighborhoods of the limit but rather, just the basic open neighborhoods of the limit.

Theorem 3.3.2. *Let (X, τ) be a topological space and suppose $\mathcal{B}$ is a base for τ. Let $(x_n)_{n=1}^{\infty}$ be a sequence in X and suppose $x \in X$. The sequence $x_n \xrightarrow{\tau} x$ if and only* *if, for every basic open neighborhood U of x, there exists $N \in \mathbb{Z}_+$ such that, for all $n \geq N$, we have $x_n \in U$.*

Proof. For the forward direction, suppose $x_n \xrightarrow{\mathcal{T}} x$. Let U be a basic open neighborhood of x. Then U is an open neighborhood of x and, since $x_n \xrightarrow{\mathcal{T}} x$, there exists $N \in \mathbb{Z}_+$ such that, for all $n \geq N$, we have $x_n \in U$.

For the other direction, let O be an open neighborhood of x. By Proposition 3.1.6, there exists a basic open neighborhood U of x such that $U \subseteq O$. By our assumption, there exists $N \in \mathbb{Z}_+$ such that, for all $n \geq N$, we have $x_n \in U \subseteq O$. □

Example 3.3.3. Consider the sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}$ given by $x_n = 1 - \frac{1}{n}$.

(i) Suppose $\mathbb{R}$ has the usual topology. Then we have seen that the sets of the form (a, b), where $a < b$, form a base for the topology. Let U be a basic open neighborhood of 1 (the limit). Then $U = (a, b)$ for some $a, b \in \mathbb{R}$, where $a < 1 < b$. Let $d = \min\{1 - a, b - 1\}$. Then $1 \in (1 - d, 1 + d) \subseteq (a, b)$. Pick $N \in \mathbb{Z}_+$ such that $N > \frac{1}{d}$. Then, for all $n \geq N$, we have that

$$x_n = 1 - \frac{1}{n} \geq 1 - \frac{1}{N} > 1 - d.$$

Clearly, $x_n < 1 < 1 + d$ and so $x_n \in (1 - d, 1 + d) \subseteq (a, b)$. Therefore, $x_n \to 1$ in the usual topology.

(ii) Now suppose $\mathbb{R}$ has the left-hand topology. Thus, sets of the form $[a, b)$, where $a < b$, form a base for the topology. In this case, the sequence $(x_n)_{n=1}^{\infty}$ does **not** converge to 1 (or anything else, for that matter) since, for example $[1, 2)$ is a basic open neighborhood of 1 yet $x_n \notin [1, 2)$ for all $n \in \mathbb{Z}_+$.

(iii) Suppose now that $\mathbb{R}$ has the discrete topology. Then the sequence $(x_n)_{n=1}^{\infty}$ does **not** converge to 1 (or anything else, for that matter). The proof is an exercise below.

(iv) Finally, suppose $\mathbb{R}$ has the indiscrete topology. Then, $x_n \to x_0$ for every real number x_0. The proof is an exercise below.

Exercise 3.3.4. Consider the sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}$ given by $x_n = 1 - \frac{1}{n}$.

(i) Suppose $\mathbb{R}$ has the discrete topology. Prove $(x_n)_{n=1}^{\infty}$ does not converge to 1.

(ii) Suppose $\mathbb{R}$ has the indiscrete topology and let $x_0 \in \mathbb{R}$. Prove $x_n \to x_0$.

While examples (ii) and (iii) above might seem a little strange, we are certainly comfortable with the idea of sequences which do not converge. What, perhaps, is very uncomfortable is what happens in example (iv). Not only have we probably never seen a sequence which converged to two different limits, but in example (iv), the sequence converges to uncountably many limits! This is one of the many reasons to explore various properties of topologies and classify them into different types. We would like to know, for example, which types of topologies can have sequences which converge to multiple limits and which types do not. We will discuss these properties in a later chapter but it is worth pointing out here.

Proposition 3.3.5. *Let X be a nonempty set and let τ_1 and τ_2 be topologies on X where τ_1 is weaker than τ_2. Let $(x_n)_{n=1}^{\infty}$ be a sequence in X and let $x \in X$. If $x_n \xrightarrow{\tau_2} x$, then $x_n \xrightarrow{\tau_1} x$.*

Proof. Let $(x_n)_{n=1}^{\infty}$ be a sequence in X such that $x_n \xrightarrow{\tau_2} x$, for some $x \in X$. We want to show $x_n \xrightarrow{\tau_1} x$. To this end, let $O \in \tau_1$ such that $x \in O$. Since $\tau_1 \subseteq \tau_2$, we have that $O \in \tau_2$. Since $x_n \xrightarrow{\tau_2} x$, there exists $N \in \mathbb{Z}_+$ such that, for all $n \geq N$, we have that $x_n \in O$. Thus, $x_n \xrightarrow{\tau_1} x$. □

Exercise 3.3.6. Give an example of a set X, topologies τ_1 and τ_2, where $\tau_1 \subseteq \tau_2$, along with a sequence $(x_n)_{n=1}^{\infty}$ in X such that $x_n \xrightarrow{\tau_1} x$, for some $x \in X$, but the sequence $(x_n)_{n=1}^{\infty}$ does not converge to x with respect to τ_2.

The above proposition and exercise help to explain the language being used when we say that the topology τ_1 is *weaker* than τ_2 as they show that convergence with respect to the weaker topology is a weaker condition than convergence with respect to the stronger topology.

3.4 Continuous Functions Between Topological Spaces

Definition 3.4.1. Let (X, τ) and (Y, σ) be topological spaces and let $f : (X, \tau) \to (Y, \sigma)$. We say f is **continuous at** $x_0 \in X$ if, for every open neighborhood U of $f(x_0)$ in Y, there exists an open neighborhood O of x_0 in X such that $f(O) \subseteq U$. If f is continuous at x_0 for each $x_0 \in X$, then we say f is continuous on X or simply say f is continuous.

Exercise 3.4.2. Let (X, τ) and (Y, σ) be topological spaces and suppose $\mathcal{B}$ is a base for τ and $\mathcal{D}$ is a base for σ. Let $f : (X, \tau) \to (Y, \sigma)$ and let $x_0 \in X$. Suppose that for each basic open neighborhood D of $f(x_0)$ in Y, there exists a basic open neighborhood B of x_0 in X such that $f(B) \subseteq D$. Prove f is continuous at x_0.

Proposition 3.4.3. *Let (X, τ) and (Y, σ) be topological spaces and let $f : (X, \tau) \to (Y, \sigma)$. Then, f is continuous on X if and only if, for every $O \in \sigma$ we have that $f^{-1}(O) \in \tau$.*

Proof. For the forward direction, suppose f is continuous on X. Let $O \in \sigma$. Let $x \in f^{-1}(O)$. Then $f(x) \in O$ and, since f is continuous a x, there exists an open neighborhood U_x of x in τ such that $f(U_x) \subseteq O$. Hence, $U_x \subseteq f^{-1}(O)$. Then $f^{-1}(O) = \cup_{x \in f^{-1}(O)} U_x \in \tau$.

For the other direction, suppose $f^{-1}(O) \in \tau$, for all $O \in \sigma$. Let $x \in X$ and let O be an open neighborhood of $f(x)$ in Y. Then $x \in f^{-1}(O)$ and, by assuption, $f^{-1}(O)$

is open. So, $f^{-1}(O)$ is an open neighborhood of x and $f(f^{-1}(O) \subseteq O$. Hence, f is continuous at x. This holds for all $x \in X$ and so f is continuous on X. □

Exercise 3.4.4. Let (X, τ) and (Y, σ) be topological spaces and let $f : (X, \tau) \to (Y, \sigma)$. Prove that f is continuous if and only if $f^{-1}(C)$ is closed for every closed set C in Y.

Example 3.4.5. (i) Let X be any nonempty set and let τ be the discrete topology on X. Let (Y, σ) be any topological space and let $f : (X, \tau) \to (Y, \sigma)$. Then f is continuous.

(ii) Let $X = Y = \mathbb{R}$ and $\tau = \sigma$ be the usual topology on $\mathbb{R}$. Then all of your continuous functions $f : \mathbb{R} \to \mathbb{R}$ from high school and calculus (like polynomials, sin, cos, etc.) are still continuous when viewed as $f : (X, \tau) \to (Y, \sigma)$.

(iii) Let $X = Y = \mathbb{R}$, τ be the usual topology on X and σ be the left-hand topology on Y. Let $f : (X, \tau) \to (Y, \sigma)$ where $f(x) = x$ for all $x \in X$. Then f is **not** continuous. The proof is a following exercise.

Exercise 3.4.6. Let $X = Y = \mathbb{R}$, τ be the usual topology on X and σ be the left-hand topology on Y. Let $f : (X, \tau) \to (Y, \sigma)$ where $f(x) = x$ for all $x \in X$. Prove f is **not** continuous.

Similar to convergent sequences, if the topology of our codomain has a base, then it is enough to check that the inverse image of every basic open set is open in the domain.

Theorem 3.4.7. *Let (X, τ) and (Y, σ) be topological spaces and suppose $\mathcal{B}$ is a basis for σ. Let $f : (X, \tau) \to (Y, \sigma)$. Then f is continuous if and only if, for every $U \in \mathcal{B}$, we have $f^{-1}(U) \in \tau$.*

Proof. The forward direction is trivial since $\mathcal{B} \subseteq \sigma$, so, if $B \in \mathcal{B}$ then $B \in \sigma$ and therefore we have that $f^{-1}(B) \in \tau$.

For the backwards direction, let $O \in \sigma$. Then $O = \cup i \in IU_i$ where $U_i \in \mathcal{B}$, for all $i \in I$. Then

$$f^{-1}(O) = f^{-1}\left(\cup_{i \in I} U_i\right) = \cup_{i \in I} f^{-1}(U_i) \in \tau$$

since $f^{-1}(U_i) \in \tau$ for all $i \in I$ by assumption. Therefore, f is continuous on X. □

Exercise 3.4.8. You might be tempted in the above theorem to also incorporate a base for τ. Explain why the statement below is false.

Fake Theorem: Let (X, τ) and (Y, σ) be topological spaces and suppose $\mathcal{B}$ is a base for σ and $\mathcal{C}$ is a base for X. Let $f : (X, \tau) \to (Y, \sigma)$. Then f is continuous if and only if, for every $U \in \mathcal{B}$, we have $f^{-1}(U) \in \mathcal{C}$.

Proposition 3.4.9. *Let X and Y be nonempty sets and let τ be a topology on X and σ_1 and σ_2 be topologies on Y where σ_1 is weaker than σ_2. Let $f : X \to Y$. If $f : (X, \tau) \to (Y, \sigma_2)$ is continuous, then $f : (X, \tau) \to (Y, \sigma_1)$ is continuous.*

Proof. Let $f : (X, \tau) \to (Y, \sigma_2)$ be continuous. To show $f : (X, \tau) \to (Y, \sigma_1)$ is continuous, we have to show $f^{-1}(O) \in \tau$ for all $O \in \sigma_1$. But, if $O \in \sigma_1$, then $O \in \sigma_2$ since $\sigma_1 \subseteq \sigma_2$. And, since $f : (X, \tau) \to (Y, \sigma_2)$ is continuous, we have that $f^{-1}(O) \in \tau$. □

Proposition 3.4.10. *Let X and Y be nonempty sets and let τ_1 and τ_2 be topologies on X where τ_1 is weaker than τ_2. Let σ be a topology on Y and let $f : X \to Y$. If $f : (X, \tau_1) \to (Y, \sigma)$ is continuous, then $f : (X, \tau_2) \to (Y, \sigma)$ is continuous.*

Proof. Let $f : (X, \tau_1) \to (Y, \sigma)$ be continuous. To show $f : (X, \tau_2) \to (Y, \sigma)$ is continuous, we have to show $f^{-1}(O) \in \tau_2$ for all $O \in \sigma$. Since $f : (X, \tau_1) \to (Y, \sigma)$ is continuous, we know $f^{-1}(O) \in \tau_1$ and, since $\tau_1 \subseteq \tau_2$, we have that $f^{-1}(O) \in \tau_2$. □

The above two propositions show that we have to be careful when using the language of "stronger" and "weaker" when discussing continuous functions between topological spaces. For a function $f : X \to Y$, continuity with respect to a weaker topology on Y is, in fact, a weaker condition than continuity with respect to a stronger topology on Y. However, continuity with respect to a weaker topology on X is actually a *stronger* condition than continuity with respect to a stronger topology on X.

Exercise 3.4.11. Let(X, τ), (Y, σ), and (Z, η) be topological spaces and let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be continuous. Then $g \circ f : (X, \tau) \to (Z, \eta)$ is continuous.

3.5 Homeomorphisms

If the reader has ever taken an abstract algebra class, then they are familiar with the idea of isomorphic groups. To be clear, if we have an isomorphism between two groups, then we say the groups are isomorphic. What this tells us is that, algebraically, there is really no difference between the two groups. If we just relabel the elements of one group (which is basically what the isomorphism does), then you will end up with the other group and vice versa. In topology, instead of isomorphisms, we have what are called homeomorphisms. They are a map between two topological spaces with certain properties, the existence of which tells us that the two topological spaces are topologically equivalent. That is, from a topological perspective, the two spaces are essentially the same. Let us now be more rigorous.

Definition 3.5.1. Let (X, τ) and (Y, σ) be two topological spaces and let $f : (X, \tau) \to (Y, \sigma)$. We say f is a **homeomorphism** if f is a bijection, f is continuous, and f^{-1} (which is well-defined since f is a bijection) is continuous. In this case, we say the topological spaces (X, τ) and (Y, σ) are **homeomorphic**.

Before we look at examples, the next theorem and exercise give a more clear understanding as to why two homeomorphic spaces are topologically equivalent.

Theorem 3.5.2. *Let $(X, \tau$ and (Y, σ) be topological spaces and suppose $f : (X, \tau) \to (Y, \sigma)$ is a bijection. The following are equivalent:*

(i) f is a homeomorphism,

(ii) $O \in \tau \Leftrightarrow f(O) \in \sigma$,

(iii) $U \in \sigma \Leftrightarrow f^{-1}(U) \in \tau$,

(iv) C is closed in $X \Leftrightarrow f(C)$ is closed in Y, and

(v) D is close in $Y \Leftrightarrow f^{-1}(D)$ is closed in X.

Proof. We will proceed by proving (i)⇒(ii)⇒(iii)⇒(iv)⇒(v)⇒(i).

[(i)⇒(ii)] Let f be a homeomorphism. Let $O \in \tau$. Since f^{-1} is continuous, we have that $f(O) = (f^{-1})^{-1}(O) \in \sigma$. For the other direction, suppose $f(O) \in \sigma$ for some $O \subseteq X$. Since f is cointinuous, $O = f^{-1}(f(O)) \in \tau$.

[(ii)⇒ (iii)] Let $U \in \sigma$. If $f^{-1}(U) \notin \tau$ then, by (i), we have that $U = f(f^{-1}(U) \notin \sigma$. For the other direction, let $U \subseteq Y$ and suppose $f^{-1}(U) \in \tau$. Then, by (i), we have that $U = f(f^{-1}(U)) \in \sigma$.

[(iii)⇒(iv)] Let C be a closed set in X. Then $C^c \in \tau$. If $f(C^c) \notin \sigma$ then, by (iii), we have that $C^c = f^{-1}(f(C^c)) \notin \tau$. So, we have that $f(C^c) \in \sigma$ and, since $f(C^c) = f(C)^c$, we have that $f(C)^c \in \sigma$ and so $f(C)$ is closed. For the other direction, suppose $f(C)$ is closed, where $C \subseteq X$. Then $f(C)^c \in \sigma$. By (iii), we have $f^{-1}(f(C)^c) \in \tau$ and $C^c = (f^{-1}(f(C)))^c$, so $C^c \in \tau$. Hence, C is closed in X.

[(iv)⇒(v)] Suppose D is closed in Y. If $f^{-1}(D)$ is not closed in X then, $f(f^{-1}(D)) = D$ is not closed in Y. Hence, $f^{-1}(D)$ must be closed in Y. For the other direction, suppose $f^{-1}(D)$ is closed in X. Then $f(f^{-1}(D)) = D$ is closed in Y.

[(v)⇒(i)] Let O be open in Y. Then O^c is closed in Y. By (v), we have that $f^{-1}(O^c)$ is closed in X. That is, $f^{-1}(O)^c$ is closed in X, and so $f^{-1}(O) \in \tau$. Now, let O be open in X. Then O^c is closed in X. If $f(O^c)$ is not closed in Y then, by (v), we have that $f^{-1}(f(O^c)) = O^c$ is not closed in X. Hence, we must have that $f(O^c)$ is closed in Y and so $f(O) \in \sigma$. □

The above theorem emphasizes the fact that the two spaces (X, τ) and (Y, σ) are topologically equivalent. The function f relabels the elements of X in a way which preserves the topological structure of the space. That is to say, it preserves the open sets (and therefore the closed sets as well).

Exercise 3.5.3. Let $\mathcal{T}$ be the set of all topological spaces and define a relation $\sim$ on $\mathcal{T}$ by $(X, \tau) \sim (Y, \sigma)$ if and only if (X, τ) and (Y, σ) are homeomorphic. Prove $\sim$ is an equivalence relation on $\mathcal{T}$ *Hint: We have to show that $\sim$ satisfies the defining three properties of an equivalence relation; reflexivity, symmetry, and transitivity. That is, show that any topological space is homeomorphic to itself; that if (X, τ) is homeomorphic to (Y, σ), then (Y, σ) is homeomorphic to (X, τ); and if (X, τ) is homeomorphic to (Y, σ) and (Y, σ) is homeomorphic to (Z, η), then (X, τ) is homeomorphic to (Z, η).*

Notation: Because $\sim$ defined above is an equivalence relation, from now on we will write $(X, \tau) \cong (Y, \sigma)$ if (X, τ) and (Y, σ) are homeomorphic.

Example 3.5.4. (i) Consider $\mathbb{R}^2$ with the usual topology τ (the topology induced by the usual metric, d_0, on $\mathbb{R}^2$) and $\mathbb{C}$ with the topology σ induced by the metric d_1 on $\mathbb{C}$ given by $d(x, y) = |x - y|$, where $|z|$ is the modulus of $z \in \mathbb{C}$. Then $(\mathbb{R}^2, \tau)$ is homeomorphic to $(\mathbb{C}, \sigma)$.

(ii) Let $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ where $a < b$. It is not difficult to see that all the sets of the form (c, d), where $c, d \in \mathbb{R} \cup \{-\infty, \infty\}$ and $a \leq c < d \leq b$ form a base for a topology on (a, b). If we endow all of our open intervals with this topology then, for example $(0, 1)$ is homeomorphic to $(1, \infty)$, and $(-10, 10)$ is homeomorphic to $(-1,1)$. In fact, any open interval with this topology is homeomorphic to any other open interval with its corresponding topology.

(iii) Consider the usual topology τ_2 on $\mathbb{R}$ and recall that it is the topology generated by the usual metric on $\mathbb{R}$. Now, recall the metric $d_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $d_1((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$ and let τ_1 be the topology generated by d_1. Then $(\mathbb{R}, \tau_2)$ and $(\mathbb{R}, \tau_1)$ are homeomorphic. In fact, $\tau_1 = \tau_2$! This fact will become abundantly clear later in the notes when we discuss topologies generated by norms on finite dimensional vector spaces.

We will see many other examples of homeomorphic spaces as we progress through these notes.

Index