MATH 422-Introduction to Topology

Matthew Ziemke

January 27, 2026

Contents

Chapter 1

Preliminaries

Chapter 2

Bridging the Gap: Metric Spaces to Topological Spaces

2.1 Need for Distance

In this chapter, we will discuss the motivations for defining metric spaces along with examining some basic concepts, properties, and examples of metric spaces. We do so, at this stage, to then motivate the more general concept of topological spaces which is the topic of this course. We will revisit metric spaces later in the notes, once we have a sufficient background in topology.

Starting in the late 1800s, especially due to the new (at the time) set-theoretic approach to mathematics, it became increasingly necessary to have a rigourous way to do "calculus-type stuff" with sets of objects not from $\mathbb{R}^n$, and with functions whose domains and codomains contain objects not from $\mathbb{R}^n$. For example, you may have learned in a differential equations course how to solve a system of differential equations using a series of the form $\sum_{n=1}^{\infty} A_n$ where A_n is a $k \times k$ matrix. But how do we actually define this series? Obviously, it is the limit of the sequence of partial sums but the sequence of partial sums is a sequence of $k \times k$ matrices. Also note that the limit is a $k \times k$ matrix. In fractal geometry, we are interested in defining the limits of sequences of subsets of $\mathbb{R}^n$ which turn out to be useful in processing images on computers. How do we define the limit in a way which makes sense? Also in a differential equations class or possibly in a linear algebra class, you may have discussed differentiation as a function. More precisely, we can think of differentiation as a function $D : A \to B$ where A is a set of differentiable functions on some domain and B is a set of functions with a certain domain, possibly having other properties. Continuous functions, as we all know, have very nice properties. Perhaps we want to know if D has some of these properties. Is D continuous? What does that even mean?

We learned in our Calculus classes the importance of the concepts of convergent sequences and continuous functions. These are by no means the only important concepts but we will focus on these two in order to illustrate the need for metric spaces and, later, topological spaces. Let's first examine the definitions of convergent sequences and continuous functions from Calculus.

Definition 2.1.1. Let $(x_n)_{n=1}^{\infty}$ be a real-valued sequence. We say that $(x_n)_{n=1}^{\infty}$ **converges** if there exists $L \in \mathbb{R}$ such that for all $\epsilon > 0$, there exists $N \in \mathbb{Z}_+$ such that, for all $n \geq N$, we have $|x_n - L| < \epsilon$. In this case we say $(x_n)_{n=1}^{\infty}$ converges to L and write $x_n \to L$.

Definition 2.1.2. Let A be a set and let $f : A \to \mathbb{R}$. For $x \in A$ we say f is **continuous at** x if, for all $\epsilon > 0$, there exists $\delta > 0$ such that if $y \in A$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$. If f is continuous at x for all $x \in A$, then we say f is continuous on A or, since A is the domain of f, we simply say f is continuous.

Let's examine the definitions above and think about whether or not they apply to some more general sequences or functions. For example, does the definition for a convergent sequence work if we have a sequence of $k \times k$ matrices instead of a sequence of real numbers? Let's see what it would look like.

Fake Definition: Let $(A_n)_{n=1}^{\infty}$ be a sequence of $k \times k$ matrices. We say $(A_n)_{n=1}^{\infty}$ converges if there exists a $k \times k$ matrix B such that for all $\epsilon > 0$, there exists $N \in \mathbb{Z}_+$ such that, for all $n \geq N$, we have $|A_n - B| < \epsilon$.

Does it work? The problem here is the expression "$|A_n - B| < \epsilon$." Given $k \times k$ matrices A_n and B, certainly $A_n - B$ makes sense. We learn how to subtract matrices in Linear Algebra. But what is the absolute value of a matrix? Of course, we could define it to be anything we want, as long as the end result is a real number so that the "$< \epsilon$" part still works. But how do we do it in a way that makes sense and gives us the limits that we want?

Let's instead think about the same definition but with a sequence of subsets from $\mathbb{R}^2$.

Fake Definition: Let $(C_n)_{n=1}^{\infty}$ *be a sequence of subsets of* $\mathbb{R}^2$*. We say* $(C_n)_{n=1}^{\infty}$ *converges if there exists a subset* B *of* $\mathbb{R}^2$ *such that for all* $\epsilon > 0$*, there exists* $N \in \mathbb{Z}_+$ *such that, for all* $n \geq N$*, we have that* $|C_n - B| < \epsilon$*.*

The problem here is again the inequality that shows up at the end, namely, "$|C_n - B| < \epsilon$." Again, we do not have a clear way to define the absolute value of a subset of $\mathbb{R}^2$ but we also do not have a relevant concept of substraction between two subsets of $\mathbb{R}^2$ either.

If we examine the definition of a continuous function f and try to replace $A \subseteq \mathbb{R}$ and $\mathbb{R}$ with generic sets, we will see that we run into precisely the same sort of issues. How do we make sense of "$|x - y| < \delta$" and "$|f(x) - f(y)| < \epsilon$?"

It seems that, in order for the definition of a convergent sequence of real numbers to make sense for sequences of objects converging to other objects or for the definition of a continuous function to make sense between sets of generic objects, we need the set of those objects to have an algebraic structure where subtraction is defined as well as an absolute value which associates one of these objects to a nonnegative number. If we follow this path, it will lead us to the study of mathematical objects called normed vector spaces. While they are certainly related to the discussion that we are having, they are too restrictive (the question of sequences of subsets of $\mathbb{R}^2$, for example, would still not be resolved). Instead, let's reinterpret the expression "$|x-y|$," where x and y are real numbers. Rather than thinking of it as the absolute value of x minus y, think of it as the magnitude (or length) of the difference. Upon reflection, this is precisely the *distance* between x and y. We do not *need* the concepts of subtraction and magnitude to define distance. We just need a meaningful way to define distance between these objects. That is, for example, we can say a sequence of matrices $(A_n)_{n=1}^{\infty}$ converges to a matrix B if, for all $\epsilon > 0$, there exists an $N \in \mathbb{Z}_+$ such that, for all $n \geq N$, we have that the distance between A_n and B is less than ϵ. How do we define distance in a meaningful and useful way? We use something called a metric.

2.2 Brief Introduction to Metric Spaces

Definition 2.2.1. A **metric** on a set X is a function $d : X \times X \to \mathbb{R}$ satisfying, for $x, y, z \in X$:

(i) $d(x, y) \geq 0$,

(ii) $d(x, y) = 0$ if and only if $x = y$,

(iii) $d(x, y) = d(y, x)$, and

(iv) $d(x, y) \leq d(x, z) + d(z, y)$.

If X is a set and d is a metric on X, then we call the ordered pair (X, d) a **metric space**.

Note that we think of the expression "$d(x, y)$" as the distance between x and y and it is a perfectly healthy interpretation. In this light, the defining properties of a metric make a lot of sense. Property (i) says that the distance between two objects should be a nonnegative real number. I certainly cannot envision a scenario where negative distance or complex-valued distance would make sense. Property (ii) says that the distance between an object and itself should be zero and that the distance between different objects should not be zero. Surely, some strange consequences would arise if neither were the case. Property (iii) tells us that if we measure the distance from x to y and then measure the distance from y to x, we should get the same thing. Lastly, Property (iv) tells us, with a little thought, that the "shortest distance between two points is a straight line." Or, said differently, while travelling from x to y, the fastest route is to go straight from x to y. If we take a detour along the way to z then it will make the trip longer (or possibly the same, if z was already on our way).

Example 2.2.2. Let's look at a few examples of metric spaces. We will see more examples later in the notes.

(i) Let $X = \mathbb{R}$ and define $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $d(x, y) = |x - y|$. Then $(\mathbb{R}, d)$ is a metric space. We call d the **usual metric on** $\mathbb{R}$. More generally, let $n \in \mathbb{Z}_+$ and let $X = \mathbb{R}^n$. Define $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$d(\overline{x}, \overline{y}) = \sqrt{\sum_{k=1}^{n} |x_k - y_k|^2}$$

where $\overline{x} = (x_1, x_2, \ldots, x_n)$ and $\overline{y} = (y_1, y_2, \ldots, y_n)$. Here, we call d the **usual metric on** $\mathbb{R}^n$ or the **Euclidean metric on** $\mathbb{R}^n$. Note that this is the metric we use in Euclidean geometry and multivariate calculus. Checking that d satisfies Properties (i) through (iii) is trivial. Checking Property (iv) requires some work, which we will omit at this time to avoid getting sidetracked.

(ii) Let $X = \mathbb{R}^2$ and define $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|.$$

Then d is a metric on $\mathbb{R}^2$ called the **taxi-cab metric**. Note that it is not the same metric given in (i) above. The fact that d is a metric and is different from the one given in (i) is a following exercise.

(iii) Let $M_n(\mathbb{R})$ be the set of all $n \times n$ matrices with real coefficients and define $d : M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to \mathbb{R}$ by

$$d((a_{ij})_{i,j=1}^{n}, (b_{ij})_{i,j=1}^{n}) = \max\{|a_{ij} - b_{ij}| : i, j = 1, \ldots, n\}.$$

Then d is a metric on $M_n(\mathbb{R})$.

Proof. *in class* □

(iv) Let X be any set and define $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}.$$

Then d is a metric on X.

Proof. in class □

Exercise 2.2.3. Let d_1 be the taxi-cab metric on $\mathbb{R}^2$ and let d_2 be the Euclidean metric on $\mathbb{R}^2$.

(i) Prove d_1 is a metric on $\mathbb{R}^2$.*Hint: You will need to use the triangle inequality which states that, for all $x, y, z \in \mathbb{R}$, $|x - y| \leq |x - z| + |z - y|$. You do not need to prove the triangle inequality.*

(ii) Give an example which shows that $d_1 \neq d_2$. That is, find $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$ such that $d_1((x_1, x_2), (y_1, y_2)) \neq d_2((x_1, x_2), (y_1, y_2))$.

Exercise 2.2.4. Let (X, d) be a metric space. Let $\alpha \in (0, \infty)$ and define a new metric $d_\alpha : X \times X \to \mathbb{R}$ by $d_\alpha(x, y) = \alpha d(x, y)$. Prove that (X, d_α) is a metric space.

2.3 Sequences and Continuous Functions for Metric Spaces

We can now examine the definitions of convergent sequences in metric spaces and continuous functions between metric spaces.

Definition 2.3.1. Let (X, d) be a metric space and let $(x_n)_{n=1}^{\infty}$ be a sequence in X. For $x \in X$, we say the sequence $(x_n)_{n=1}^{\infty}$ **converges to** x if, for all $\epsilon > 0$, there exists $N \in \mathbb{Z}_+$ such that, for all $n \geq N$, we have $d(x_n, x) < \epsilon$. We write $x_n \to x$, if the metric we're using is understood, or $x_n \xrightarrow{d} x$ to specify the metric being used.

If we take $X = \mathbb{R}$ and d to be the usual metric on $\mathbb{R}$ then the above definition is precisely the definition for a convergent sequence in our Calculus classes.

Exercise 2.3.2. Let X be a nonempty set and define $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases} .$$

Then d is a metric on X. Let $(x_n)_{n=1}^{\infty}$ be a sequence in X and $x \in X$ such that $x_n \xrightarrow{d} x$. Prove that there exists $N \in \mathbb{Z}_+$ such that $x_n = x$ for all $n \geq N$.

Now, let us look at the definition of a continuous function between two metric spaces.

Definition 2.3.3. Let (X, d_0) and (Y, d_1) be metric spaces and let $f : (X, d_0) \to (Y, d_1)$. We say the function f is **continuous at** x if, for all $\epsilon > 0$, there exists $\delta > 0$ such that, if $x_0 \in X$ and $d_0(x, x_0) < \delta$, then $d_1(f(x), f(x_0)) < \epsilon$. If $A \subseteq X$ and f is continuous at x for all $x \in A$, then we say f is **continuous on** A. If f is continuous on X, we often just say "f is continuous," meaning f is continuous on its entire domain.

Again, if $X = Y = \mathbb{R}$ and $d_0 = d_1$ is the usual metric on $\mathbb{R}$, then the definition above is precisely the definition from our Calculus classes. Note that we write "$f : (X, d_0) \to (Y, d_1)$" instead of simply "$f : X \to Y$" when discussing continuity since it is important which metrics are being used. The importance is illustrated in the following example.

Example 2.3.4. Consider the function $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = x$ for all $x \in \mathbb{R}$. Let d_0 be the usual metric on $\mathbb{R}$ and let d_1 be the metric defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases} .$$

(i) The function $f : (\mathbb{R}, d_1) \to (\mathbb{R}, d_0)$ is continuous on $\mathbb{R}$. Why? Fix an arbitrary $y \in \mathbb{R}$. We will then show f is continuous at y. Let $\epsilon > 0$. Pick $\delta = \frac{1}{2}$. Then, if $x \in \mathbb{R}$ and $d_1(x, y) < \delta = \frac{1}{2}$, then we must have that $d_1(x, y) = 0$ and so $x = y$. Hence,

$$d_0(f(x), f(y)) = d_0(x, y) = 0 < \epsilon$$

and so f is continuous at y. Since this holds for all $y \in \mathbb{R}$, we have that $f : (\mathbb{R}, d_1) \to (\mathbb{R}, d_0)$ is continuous on $\mathbb{R}$.

(ii) The function $f : (\mathbb{R}, d_0) \to (\mathbb{R}, d_1)$ is not continuous at y, for any $y \in \mathbb{R}$. Why? Let $\epsilon = \frac{1}{2}$. Then, for any $\delta > 0$, find an $x \in \mathbb{R}$ such that $d_0(x, y) < \delta$ but $x \neq y$ (we can take $x = y + \frac{\delta}{2}$, for example). Then, we have that $d_0(x, y) < \delta$ but

$$d_1(f(x), f(y)) = d_1(x, y) = 1 > \frac{1}{2} = \epsilon$$

and so $f : (\mathbb{R}, d_0) \to (\mathbb{R}, d_1)$ is not continuous at y for any $y \in \mathbb{R}$.

2.4 From Metrics to Open Sets

As stated earlier, we will examine metric spaces in more detail later as there is still a lot to discuss. The goal here is to find a way to generalize the concepts of convergent sequences and continuous functions to sets of objects which do not have appropriate metrics (as we've seen, we can always define a metric on a set but it might not have the properties we like or it might not reflect the physical system we're trying to model). For example, suppose our set is $\mathbb{R}^\mathbb{R}$ and we want a sequence of functions $(f_n)_{n=1}^\infty$ in $\mathbb{R}^\mathbb{R}$ to converge to some $f \in \mathbb{R}^\mathbb{R}$ if and only if $f_n(x) \to f(x)$, for all $x \in \mathbb{R}$ (using the usual metric). It can be proven that there exists no metric d on $\mathbb{R}^\mathbb{R}$ such that $(f_n)_{n=1}^\infty$ converges to a function $f \in \mathbb{R}^\mathbb{R}$ with respect to d if and only if $f_n(x) \to f(x)$ for all $x \in \mathbb{R}$. But this is an important type of convergence! In Real Analysis, we call such convergence **pointwise convergence**. So, once again, we need to reexamine the definitions at the beginning of the chapter to see if we can find a way to generalize them to sequences in sets of objects which do not have a metric as well as functions between sets of objects which do not have a metric. In order to do this, we will have to find equivalent statements to both definitions first. This will involve the concepts of open and closed sets.

Definition 2.4.1. Let (X, d) be a metric space and let $x \in X$. For $r > 0$, we define the **open ball centered at** x **with radius** r to be the set

$$B_d(x, r) = \{y \in X : d(x, y) < r\}.$$

Example 2.4.2. (i) If d is the usual metric on $\mathbb{R}$, then $B_d(0, 5) = (-5, 5)$ and $B_d(4, 1) = (3, 5)$.

(ii) If d is the usual metric on $\mathbb{R}^2$, then $B_d((1, 2), 1)$ is the open disk centered at $(1, 2)$ with radius 1.

(iii) If d is the taxicab metric on $\mathbb{R}^2$, then $B_d((0, 0), 1)$ is an open square with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$.

(iv) If X is any nonempty set and d is the metric where $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ if $x \neq y$, then, for any $x \in X$, $B_d(x, \frac{1}{2}) = \{x\}$, while $B_d(x, 2) = X$.

Definition 2.4.3. Let (X, d) be a metric space. We say a subset $O \subseteq X$ is **open** if, for all $x \in O$, there exists $r > 0$ such that $B_d(x, r) \subseteq O$. Note that this definition implies $\emptyset$ and X are open sets.

Exercise 2.4.4. Let (X, d) be a metric space. Let $x \in X$ and $r > 0$. Prove that $B_d(x, r)$ is an open set. *Hint: Let $y \in B_d(x, r)$. Find $r_0 > 0$ such that $B_d(y, r_0) \subseteq B_d(x, r)$.*

Exercise 2.4.5. Let d be the usual metric on $\mathbb{R}$. Let $a, b \in \mathbb{R}$, where $a < b$.

(i) Prove the interval (a, b) is an open set.

(ii) Prove the interval $[a, b)$ is not an open set.

(iii) Prove the interval $(a, b]$ is not an open set.

(iv) Prove the interval $[a, b]$ is not an open set.

Theorem 2.4.6. *Let (X, d) be a metric space.*

(i) Let I be an index set and suppose O_i is an open set, for all $i \in I$. Then $\cup_{i\in I} O_i$ is an open set.

(ii) Let $n \in \mathbb{Z}_+$ and suppose $O_1, O_2, \ldots, O_n$ are open sets. Then $\cap_{i=1}^n O_i$ is an open set.

Proof. To prove (i), let I be an index set and suppose O_i is open, for all $i \in I$. Let $x \in \cup_{i\in I} O_i$. Then there exists $k \in I$ such that $x \in O_k$. Since O_k is open, there exists $r > 0$ such that $B_d(x, r) \subseteq O_k$. Then

$$B_d(x, r) \subseteq O_k \subseteq \cup_{i\in I} O_i$$

and so $\cup_{i\in I} O_i$ is open.

To prove (ii), let $n \in \mathbb{Z}_+$ and suppose $O_1, O_2, \ldots, O_n$ are open sets. Let $x \in \cap_{i=1}^n O_i$. Then $x \in O_i$ for all $i = 1, 2, \ldots, n$. Since each O_i is open, there exists $r_i > 0$ such that $B_d(x, r_i) \subseteq O_i$. Let $r = \max\{r_1, r_2, \ldots, r_n\}$. Then, for each $i = 1, 2, \ldots, n$, we have $B_d(x, r) \subseteq B_d(x, r_i) \subseteq O_i$ and so $B_d(x, r) \subseteq \cap_{i=1}^n O_i$. □

Exercise 2.4.7. Let d be the usual metric on $\mathbb{R}$. Give an example of open sets $O_1, O_2, \ldots$ such that $\cap_{i=1}^\infty O_i$ is not open.

It is worth noting here that if (X, d) is a metric space and O is an open subset of X then O can always be written as a union of elements from the set

$$\mathcal{B} = \{B_d(x, r) : x \in X \text{ and } r > 0\}.$$

Indeed, for $x \in O$, since O is open, there exists $r_x > 0$ such that $B_d(x, r_x) \subseteq O$. Then $O = \cup_{x \in O} B_d(x, r_x)$. Since every open set in X can be written as a union of elements from $\mathcal{B}$, we call $\mathcal{B}$ a **base for the topology on** X and call the elements of $\mathcal{B}$ **basic open sets**.

Now, let us revisit our definitions for convergent sequences in metric spaces and continuous functions between metric spaces. First, we can rewrite both definitions using our new notation. We will present the equivalent definitions as theorems to avoid confusion.

Theorem 2.4.8. *Let (X, d) be a metric space. A sequence $(x_n)_{n=1}^{\infty}$ in X converges to $x \in X$ if and only if, for all $\epsilon > 0$, there exists $N \in \mathbb{Z}_+$ such that, if $n \geq N$, then $x_n \in B_d(x, \epsilon)$.*

Proof. The proof is obvious since $x_n \in B_d(x, \epsilon)$ if and only if $d(x_n, x) < \epsilon$. □

Theorem 2.4.9. *Let (X, d_0) and (Y, d_1) be metric spaces and let $f : (X, d_0) \to (Y, d_1)$. The function f is continuous at x, for some $x \in X$, if and only if, for all $\epsilon > 0$, there exists $\delta > 0$ such that, $f(B_{d_0}(x, \delta)) \subseteq B_{d_1}(f(x), \epsilon)$. Further, the function f is continuous if, for all $x \in X$ and for all $\epsilon > 0$, there exists $\delta_{x,\epsilon}$ such that $f(B_{d_0}(x, \delta_{x,\epsilon})) \subseteq B_{d_1}(f(x), \epsilon)$.*

Now, let's examine some other equivalent definitions for convergent sequences and continuous functions for metric spaces.

Theorem 2.4.10. *Let (X, d) be a metric space. A sequence $(x_n)_{n=1}^{\infty}$ in X converges to $x \in X$ if and only if, for all open sets O such that $x \in O$, there exists $N \in \mathbb{Z}_+$ such that, if $n \geq N$, then $x_n \in O$.*

Proof. For the forward direction, let O be an open set such that $x \in O$. Since O is open, there exists $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq O$. Since $x_n \xrightarrow{d} x$, there exists $N \in \mathbb{Z}_+$ such that, for all $n \geq N$, we have that $x_n \subseteq B_d(x, \epsilon) \subseteq O$.

For the other direction, we assume that for all open sets O, such that $x \in O$, there exists $N \in \mathbb{Z}_+$ such that, for all $n \geq N$, we have $x_n \in O$. Let $\epsilon > 0$. Since $B_d(x, \epsilon)$ is an open set, we can find $N \in \mathbb{Z}_+$ such that, for all $n \geq N$, we have $x_n \in B_d(x, \epsilon)$. □

Theorem 2.4.11. *Let (X, d_0) and (Y, d_1) be metric spaces and let $f : (X, d_0) \to (Y, d_1)$. The function f is continuous at x, for some $x \in X$, if and only if, for every open subset O of Y such that $f(x) \in O$, there exists an open subset $U_{x,O}$ of X such that $f(U_{x,O}) \subseteq O$. The function f is continuous if and only if, for every open subset O of Y, $f^{-1}(O)$ is an open subset of X.*

Proof. To prove the forward direction of the first statement, let O be an open set in Y such that $f(x) \in O$. Since O is open, there exists $\epsilon > 0$ such that $B_{d_1}(f(x), \epsilon) \subseteq O$. Since f is continuous at x, there exists $\delta_{x,\epsilon} > 0$ such that $f(B_{d_0}(x, \delta_{x,\epsilon})) \subseteq B_{d_1}(f(x), \epsilon) \subseteq O$ and note that $B_{d_0}(x, \delta_{x,\epsilon})$ is open so let $U_{x,\epsilon} = B_{d_0}(x, \delta_{x,\epsilon})$.

For the other direction of the first statement, we know that $B_{d_1}(f(x), \epsilon)$ is open. So, there exists an open set $U_{x,\epsilon}$ containing x such that $f(U_{x,\epsilon}) \subseteq B_{d_1}(f(x), \epsilon)$. Since $U_{x,\epsilon}$ is open and $x \in U_{x,\epsilon}$, there exists $\delta > 0$ such that $B_{d_0}(x, \delta) \subseteq U_{x,\epsilon}$. Then, $f(B_{d_0}(x, \delta)) \subseteq f(U_{x,\epsilon}) \subseteq O$. Hence, f is continuous at x.

For the forward direction of the second statement, suppose f is continuous on X. Let O be an open set in Y. Since f is continuous on X, f is continuous at x, for all $x \in f^{-1}(O)$. Hence, for each $x \in f^{-1}(O)$, there exists an open set $U_{x,O}$ containing x such that $f(U_{x,O}) \subseteq O$. Since

$$f^{-1}(O) = \bigcup_{x \in f^{-1}(O)} U_{x,O}$$

which is a union of open sets, we have that $f^{-1}(O)$ is open.

For the other direction of the second statement, let $x \in X$. Let O be an open subset of Y such that $f(x) \in O$. By our assumption $f^{-1}(O)$ is an open subset of X. Since

$x \in X$ and $f(f^{-1}(O) \subseteq O$, by the first statement we have that f is continuous at x. Since this is true for all $x \in X$, we then have that f is continuous on X. $\square$

Notice that the statements in both theorems above no longer require a metric but rather open sets. It is true that we used metrics to define open sets but we don't have to. Recall at the beginning of the chapter when we decided that we did not need subtraction and absolute values to define distance if we just "skipped ahead" and defined distance directly. We can do the same thing here. We do not need metrics to define open sets. We can just skip ahead and define the open sets directly. This jump in abstraction is how we arrive at topological spaces.

Index