MATH 422-Introduction to Topology

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Contents

Chapter 1

Preliminaries

1.1 Sets

Definition 1.1.1. A **set** is a collection of objects. A set is only concerned with membership. That is, a set's only concern is whether an object is a member of the set or not. If A is a set and x is an object, we say x is an **element** of A if x is a member of A and write $x \in A$. If x is not a member of A, then we say x is not an element of A and write $x \notin A$. The set with no elements is called the **empty set** and is denoted by $\emptyset$.

The above definition of a set will be satisfactory for us in these notes but it should be stated that, in general, we need to be more careful about how we define sets or else we run into issues. For example, examine the following exercise.

Exercise 1.1.2. (Russell's Paradox) Obviously, there are many examples of sets which do not contain themselves (take, for example $A = \{1, 2, 3\}$). First, convince yourself that it is possible for a set to contain itself. That is, provide an example of a set B such that $B \in B$. Next, consider the set R of all sets which do not contain themselves (the example A above would be an element of R). Explain why the question, "Is $R \in R$?" does not have a satisfactory answer.

To avoid such paradoxes, we have developed a set of axioms, or rules that must be followed, when defining sets called the Zermelo-Fraenkel Axioms (or ZF Axioms). When we also include another axiom called the Axiom of Choice (more on this later), then they are referred to as the ZFC Axioms. Since we will not need to be so careful with how we define sets, we will not examine the ZF axioms. We will, however, discuss the Axiom of Choice later.

Definition 1.1.3. Let A and B be sets. We say A is a **subset** of B, and write $A \subseteq B$, if every element of A is also an element of B. If $A \subseteq B$ and $B \subseteq A$, then we say A **equals** B, and write $A = B$.

The above definition tells us that two sets are equal precisely when they have the exact same elements. It also solidifies the idea that sets are only concerned with membership and that added structure is needed if we wish to consider concepts such as order or repetition. For example, $\{1, 1, 2, 2\} = \{1, 2\}$ since every element on the lefthand set is an element on the right, while every element in the righthand set is an element on the left. And so, sets do not care about repetition. For this reason, we would never actually write something like $\{1, 1, 2, 2\}$ since it is confusing and a waste of time. Similarly, sets do not care about order since $\{1, 2\} = \{2, 1\}$. Our most commonly used sets of numbers have standard notation given below:

(i) the set of all natural numbers, that is, the set of all nonnegative integers, is denoted by $\mathbb{N}$,

(ii) the set of all integers is denoted by $\mathbb{Z}$,

(iii) the set of all positive integers less than or equal to n, where n is a positive integer, is denoted by $\mathbb{Z}_n$,

(iv) the set of all rational numbers is denoted by $\mathbb{Q}$,

(v) the set of all real numbers is denoted by $\mathbb{R}$, and

(vi) the set of all complex numbers is denoted by $\mathbb{C}$.

We will denote the set of all positive integers by $\mathbb{Z}_+$. Given a set A, we define the **power set** of A, denoted by $\mathcal{P}(A)$ to be the set of all subsets of A.

Definition 1.1.4. Let A and B be sets. The **union** of A and B, denoted $A \cup B$, is the set of all objects x such that $x \in A$ or $x \in B$ (or both). The **intersection** of A and B, denoted $A \cap B$, is the set of all objects x such that $x \in A$ and $x \in B$. We say the sets A and B are **disjoint** if $A \cap B = \emptyset$. The set A **minus** B, or A **throw away** B, denoted by $A \setminus B$, is the set of all objects in A that are not in B. Although usually implicitly understood, often times we need to specify the **universal set**, or all the objects under consideration. This is especially true if we want to consider the **complement** of a set. That is, the complement of the set A, denoted by A^c, is the set of all object in the universal set that are not elements of A.

A couple of things worth noting here. First, the "or both" included in the definition of the union of two sets is not actually needed since "or" is always the inclusive "or" in mathematics. Second, we usually do not specify the universal set since it is contextually obvious in most cases. Nevertheless, it is technically needed to have a rigorous definition for the complement of a set. For example, if A is the set of all positive even integers, then most of us would consider A^c to be the set of all positive odd integers. This assumes though that the universal set is the set of positive integers. Without specifying the universal set, we could also have that $\sqrt{2} \in A^c$ or $-7 \in A^c$. We often refer to union, intersection, and throwaway as rank-2 operations as they give us a way to produce one set from two. For these reasons, we borrow language from algebra to say that unions and intersections are commutative, meaning $A \cup B = B \cup A$ and $A \cap B = B \cap A$, as well as associative, i.e., $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$. In the next definition, we extend our definitions of unions and intersections to collections of sets.

Definition 1.1.5. Let I be an index set and let A_i be a set for all $i \in I$. Then

$$\bigcup_{i \in I} A_i = \{x | \, x \in A_i \text{ for some } i \in I\}$$

and

$$\bigcap_{i \in I} A_i = \{x | \, x \in A_i \text{ for all } i \in I\} .$$

If the index set I is finite, say I has n elements, then there is no harm in assuming $I = \{1, 2, \ldots, n\}$ and we often write

$$\bigcup_{i \in I} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

and

$$\bigcap_{i\in I} A_i = A_1 \cap A_2 \cap \cdots \cap A_n.$$

Example 1.1.6. Let $A_n = [-\frac{1}{n}, \frac{1}{n}]$ for all $n \in \mathbb{Z}_+$, where $[a,b] = \{x \in \mathbb{R} | a \leq x \leq b\}$. Then,

$$\bigcup_{n\in\mathbb{Z}_+} A_n = (-1,1)$$

while

$$\bigcap_{n\in\mathbb{Z}_+} A_n = \{0\}.$$

Exercise 1.1.7. (DeMorgan's Laws)

(a) Let A and B be sets. Prove that $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$.

(b) Let I be an index set and let A_i be a set, for all $i \in I$. Prove that

$$\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} A_i^c \quad \text{and} \quad \left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} A_i^c.$$

Hint: Prove all of the set equalities above by "set inclusion." That is, to show sets A and B are equal, show $A \subseteq B$ and $B \subseteq A$. To do this, prove two statements: if $x \in A$, then $x \in B$; if $x \in B$, then $x \in A$.

Definition 1.1.8. Given two sets A and B, the **Cartesian product** of A and B, denoted by $A \times B$, is the set of all ordered pairs (a,b) such that $a \in A$ and $b \in B$.

Note that we call ordered pairs "ordered" pairs because we care about the order. We say $(a,b) = (c,d)$ if and only if $a = c$ and $b = d$. For this reason, $(1,2) \neq (2,1)$.The reader is probably most familiar with the Cartesian product $\mathbb{R} \times \mathbb{R}$, which we typically denote by $\mathbb{R}^2$, and visualize as the Cartesian plane. Perhaps some of the readers have taken a graph theory class or combinatorics class where you studied $\mathbb{Z} \times \mathbb{Z}$ and referred to it as the "integer lattice." There is nothing preventing us from taking Cartesian products of completely different sets though. For example, $\mathbb{Z} \times \mathbb{R}$ would be the set of all ordered pairs whose first coordinate is an integer while the second coordinate is any real number. Notice though that $\mathbb{Z} \times \mathbb{R} \neq \mathbb{R} \times \mathbb{Z}$ and so the Cartesian product is not, in general, commutative.

Continuing in this fashion, for $n \in \mathbb{Z}_+$ and sets $A_1, A_2, \ldots, A_n$, we define $A_1 \times A_2 \times \cdots \times A_n$ to be the set of all n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_i \in A_i$ for $i = 1, \ldots, n$. If $A = A_i$ for $i = 1, \ldots, n$, then we often write $A_1 \times A_2 \times \cdots \times A_n = A^n$ and so, for example, $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^3$. Note that technically, $\mathbb{R}^3 \neq \mathbb{R} \times \mathbb{R}^2 \neq \mathbb{R}^2 \times \mathbb{R}$ since the elements of $\mathbb{R}^3$ are ordered triples while the elements of $\mathbb{R} \times \mathbb{R}^2$ and $\mathbb{R}^2 \times \mathbb{R}$ are ordered pairs where one of their coordinates is also an ordered pair (for example, elements of $\mathbb{R} \times \mathbb{R}^2$ look like $(a, (b, c))$). Nevertheless, we often write things such as $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2$ since both sets are made up of ordered 3-element subsets of $\mathbb{R}$ and are essentially the same thing. We will return to discussing Cartesian products in a later section.

One common mistake students often make when first learning about Cartesian products is illustrated in the next exercise.

Exercise 1.1.9. Let A and B be sets.

(a) Let $C \subseteq A$ and $D \subseteq B$. Prove that $C \times D \subseteq A \times B$.

(b) Give an example of a set E such that $E \subset A \times B$ but $E \neq C \times D$ for any $C \subseteq A$ and any $D \subseteq B$.

To be clear about the mistake the previous example is supposed to illuminate, often times students will write things such as, "Let $E \subseteq A \times B$. Then $E = C \times D$, for some $C \subseteq A$ and some $D \subseteq B$." It should now be clear that this is not necessarily true.

1.2 Relations

Definition 1.2.1. We say a set R is a **relation** from the elements of a set A to the elements of a set B if $R \subseteq A \times B$. If $(a, b) \in R$, then we often write aRb or $a \sim b$. If R is a relation from the elements of a set A to the elements of the same set A then we simply say that R is a relation on A. If R is a relation on a set A, we say the relation is

(i) **reflexive** if aRa for all $a \in A$,

(ii) **symmetric** if aRb whenever bRa, and

(iii) **transitive** if, whenever aRb and bRc, we have that aRc.

An example of a relation on $\mathbb{R}$ would be $<$ where we say $(a, b) \in R$ if $a < b$. In this case, R would not be reflexive or symmetric but it would be transitive. A different relation on $\mathbb{R}$ would be $=$ where $(a, b) \in R$ if $a = b$. In this case, R would be reflexive, symmetric, and transitive.

Definition 1.2.2. A relation R on a set A is called an **equivalence relation** if R is reflexive, symmetric, and transitive. In this case, we often write $a \equiv b$ instead of aRb. If R is an equivalence relation on a set A then, for every $a \in A$, we define the **equivalence class** of a to be $[a] = \{x \in A | aRx\}$.

Exercise 1.2.3. Let R be an equivalence relation on a set A. Let $a, b \in A$. Prove that if aRb, then $[a] = [b]$ and if $(a, b) \notin R$, then $[a] \cap [b] = \emptyset$.

Definition 1.2.4. We say a relation R on a set A is **antisymmetric** if, whenever aRb and bRa, we must have that $a = b$.

Definition 1.2.5. We say a set A is a **partially ordered set (or poset)** if there is a relation on A which is reflexive, transitive, and antisymmetric.

An example of a partially ordered set would be $\mathcal{P}(\mathbb{N})$ where we say that, for $A, B \in \mathcal{P}(\mathbb{N})$, ARB if $A \subseteq B$. Notice here that not all of the elements of $\mathcal{P}(\mathbb{N})$ are comparable with this relation. For example, if $A = \{1, 2\}$ and $B = \{1, 3\}$ then A does not relate to B and B does not relate to A. In otherwords, $(A, B) \notin R$ and $(B, A) \notin R$. This leads us to the next definition.

Definition 1.2.6. We say a set A is **totally ordered** if there is a relation R on A which is reflexive, transitive, and antisymmetric such that, for all $a, b \in A$, either aRb or bRa (or both).

The set $\mathcal{P}(\mathbb{N})$ ordered by $\subseteq$ is not a totally ordered set by the example given above. An example of a totally ordered set would be $\mathbb{R}$ ordered by $\leq$. Obviously, this relation is reflexive and transitive. It is antisymmetric since, for any $a, b \in \mathbb{R}$, if $a \leq b$ and $b \leq a$, then $a = b$.

Definition 1.2.7. Let A be a partially ordered set where we denote the relation by $\preceq$. We say an element $a_0 \in A$ is a **minimal element** of A if, $a \preceq a_0$ implies $a = a_0$ for any $a \in A$. We say an element $a_1 \in A$ is a **maximal element** of A, if $a_1 \preceq a$ implies $a = a_1$ for any $a \in A$.

Note that minimal and maximal elements of a partially ordered set need not be unique, as the next exercise asks the reader to verify.

Exercise 1.2.8. Give an example of a partially ordered set which has more than one maximal element. That is, give an example of a partially ordered set A, along with a partial order $\preceq$ on the set A, and provide two elements $a, b \in A$ such that a and b are both maximal elements yet $a \neq b$.

Definition 1.2.9. Let A be a partially ordered set ordered by $\preceq$. We say $B \subseteq A$ is a **chain** in A if B with the partial order $\preceq$ is a totally ordered set. We say an element $a \in A$ is an **upper bound** for the chain B, if $b \preceq a$ for all $b \in B$.

For example, as we saw earlier, $\mathcal{P}(\mathbb{N})$ is a partially ordered set when it is ordered using set-inclusion. It is not a totally ordered set. If we let

$$B = \{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}\}$$

then B is a chain in $\mathcal{P}(\mathbb{N})$ since all of the elements of B are comparable. We would also say that the chain B has a maximal element $\{1,2,3\}$. So, while $\{1,2,3\}$ is not a maximal element of $\mathcal{P}(\mathbb{N})$, it is a maximal element for the chain B. We would also say that $\{1,2,3,4\} \in \mathcal{P}(\mathbb{N})$ is an upper bound for the chain B. Note that upper bounds of chains do not need to be elements of the chain.

1.3 Functions

Definition 1.3.1. Let A and B be sets. We say $f \subseteq A \times B$ is a **function** with domain A and codomain B, and write $f : A \to B$, if

(i) for all $a \in A$, there exists $b \in B$ such that $(a, b) \in f$, and

(ii) if $(a, b), (a, c) \in f$ then $b = c$.

If $(a, b) \in f$, then we often say that a **maps** to b. Also, if $(a, b) \in f$ then we often write $f(a) = b$. Note that (i) above means that every element of A must map to some element of B and (ii) means that no element of A can map to two different elements of B. In our high school classes, we often present functions as rules which tell us how to find the $f(a)$ that a maps to. We then define the **graph** of a function f to be the set of all elements of the form $(a, f(a))$. Here, we make no distinction between a function and its graph, thus eliminating a need for such a rule.

Definition 1.3.2. Let $f : A \to B$ be a function. We say f is **injective** or **one-to-one** if $(a_0, b), (a_1, b) \in f$ implies $a_0 = a_1$. We say f is **surjective** or **onto** if, for every

$b \in B$ there exists $a \in A$ such that $(a, b) \in f$. A function is said to be **bijective** if it is injective and surjective.

Example 1.3.3. If we define $f : \mathbb{N} \to \mathbb{N}$ by $f = \{(a, a^2) | a \in \mathbb{N}\}$, then f is injective since $(a_0, b), (a_1, b) \in f$ implies that $a_0 = \sqrt{b} = a_1$ but f is not surjective since $2 \in \mathbb{N}$ yet there exists no $a \in \mathbb{N}$ such that $(a, 2) \in f$.

Exercise 1.3.4. Give an example of a function$f : \mathbb{N} \to \mathbb{N}$ which is surjective but not injective.

Proposition 1.3.5. *Let $f : A \to B$ and $g : B \to C$. Define $g \circ f$ to be the set of all elements $(a, c) \in A \times C$ such that there exists $b \in B$ where $(a, b) \in f$ and $(b, c) \in g$. Then $g \circ f$ is a function with domain A and codomain C.*

Proof. First, we have to prove that for all $a \in A$, there exists $c \in C$ such that $(a, c) \in g \circ f$. Let $a \in A$. Since f is a function with domain A, there exists $b \in B$ such that $(a, b) \in f$. Since g is a function with domain B and $b \in B$, there exists $c \in C$ such that $(b, c) \in g$. Then, by definition of $g \circ f$, we have that $(a, c) \in g \circ f$.

Next, we have to prove that if $(a, c_1), (a, c_2) \in g \circ f$, then $c_1 = c_2$. Since $(a, c_1), (a, c_2) \in g \circ f$, there exists $b_1, b_2 \in B$ such that $(a, b_1) \in f$ and $(b_1, c_1) \in g$ and $(a, b_2) \in f$ and $(b_2, c_2) \in g$. Since $(a, b_1), (a, b_2) \in f$ and f is a function, we must have that $b_1 = b_2$. Then, we have that $(b_1, c_1), (b_1, c_2) \in g$. Since g is a function, we must have that $c_1 = c_2$. Therefore, $g \circ f : A \to C$. $\square$

Exercise 1.3.6. Let $f : A \to B$ and $g : B \to C$.

(a) Prove that if f and g are injective, then $g \circ f$ is injective.

(b) Prove that if f and g are surjective, then $g \circ f$ is surjective.

(c) Conclude that if f and g are bijective, then $g \circ f$ is bijective.

Remark 1.3.7. Given $f : A \to B$ and $C \subseteq A$, we define $f(C)$ to be the subset of B containing all $b \in B$ such that there exists $c \in C$ such that $(c, b) \in f$. For $D \subseteq B$, we define $f^{-1}(D)$ to be the subset of A containing all $a \in A$ such that there exists $d \in D$ so that $(a, d) \in f$.

Consider the function $f : \mathbb{N} \to \mathbb{N}$ we considered earlier, where $f(a) = a^2$ for all $a \in \mathbb{N}$. In this case,

$$f(\{1,2,3\}) = \{1,4,9\} \quad , \quad f^{-1}(\{3,4,16\}) = \{2,4\} \quad , \quad \text{and} \quad f^{-1}(\{3,7,11\}) = \emptyset.$$

Exercise 1.3.8. Let A and B be sets and let $f : A \to B$. Let $C_1, C_2 \subseteq A$ and $D_1, D_2 \subseteq B$. Fill in the following blanks with either "$\subseteq$", "$\supseteq$", "$=$," or "$\neq$."

(i) $f(C_1 \cup C_2)$ _____ $f(C_1) \cup f(C_2)$

(ii) $f(C_1 \cap C_2)$ _____ $f(C_1) \cap f(C_2)$

(iii) $f(C_1^c)$ _____ $f(C_1)^c$

(iv) $f^{-1}(D_1 \cup D_2)$ _____ $f^{-1}(D_1) \cup f^{-1}(D_2)$

(v) $f^{-1}(D_1 \cap D_2)$ _____ $f^{-1}(D_1) \cap f^{-1}(D_2)$

(vi) $f^{-1}(D_1^c)$ _____ $f^{-1}(D_1)^c$

Definition 1.3.9. Let $f : A \to B$. We define f^{-1} to be the set of all $(b, a) \in B \times A$ such that $(a, b) \in f$. If f^{-1} is a function, then we say f is **invertible** .

Let $f : A \to B$. If f is invertible, then $f^{-1} : B \to A$ and $f^{-1}(b) = a$ if and only if $f(a) = b$. If f is not invertible, then you will often still see people write $f^{-1}(b)$ but what they really mean is $f^{-1}(\{b\})$. To illustrate, consider the example of $f : \mathbb{R} \to \mathbb{R}$ where $f(a) = a^2$. It is easy to see that f^{-1} is not a function since, for example, there exists no $a \in \mathbb{R}$ such that $(-1, a) \in f^{-1}$. Hence, f is not invertible. On the otherhand, one will still see people write things like $f^{-1}(4)$ by which they actually mean $f^{-1}(4) = f^{-1}(\{4\}) = \{-2, 2\}$. In light of this, $f^{-1}(0) = \{0\}$ and $f^{-1}(-1) = \emptyset$.

Proposition 1.3.10. *Let $f : A \to B$. The function f is invertible if and only if f is a bijective.*

Proof. Suppose f is invertible. Then $f^{-1} = \{(b, a) \in B \times A |\, (a, b) \in f\}$ is a function. First, to show f is injective, suppose $(a_1, b), (a_2, b) \in f$. Then $(b, a_1), (b, a_2) \in f^{-1}$. Since f^{-1} is a function, we must have that $a_1 = a_2$. Hence, f is injective. Now, to prove f is surjective, let $b \in B$. Since f^{-1} is a function, there exists $a \in A$ such that $(b, a) \in f^{-1}$. Then $(a, b) \in f$ and so f is surjective. Hence, f is bijective.

For the other direction, suppose f is bijective. We first want to show that for all $b \in B$, there exists $a \in A$ such that $(b, a) \in f^{-1}$. Let $b \in B$. Since f is surjective, there exists $a \in A$ such that $(a, b) \in f$. Hence, $(b, a) \in f^{-1}$. Next, we want to show that if $(b, a_1), (b, a_2) \in f^{-1}$ then $a_1 = a_2$. Let $(b, a_1), (b, a_2) \in f^{-1}$. Then $(a_1, b), (a_2, b) \in f$. Since f is injective, we have that $a_1 = a_2$. Therefore, f^{-1} is a function. □

Exercise 1.3.11. Prove that if $f : A \to B$ and $g : B \to C$ are invertible, then $g \circ f$ is invertible and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

1.4 Cardinality

Definition 1.4.1. Given a collection of sets $\mathcal{C}$, define a relation $\equiv$ on $\mathcal{C}$ by $A \equiv B$ if and only if there exists a bijection $f : A \to B$. In this case, we say A has the same **cardinality** as B and write $card(A) = card(B)$ or $|A| = |B|$.

Proposition 1.4.2. *Given a collection of sets $\mathcal{C}$, the relation $\equiv$ defined above is an equivalence relation.*

Proof. Let $\mathcal{C}$ be a collection of sets.

Let $A \in \mathcal{C}$. Then $f : A \to A$ defined by $f(a) = a$ is a bijection and so $A \equiv A$.

Let $A, B \in \mathcal{C}$ and suppose $A \equiv B$. Then there exists a bijection $f : A \to B$. Then, $f^{-1} : B \to A$ is a bijection and so $B \equiv A$.

Let $A, B, C \in \mathcal{C}$ and suppose $A \equiv B$ and $B \equiv C$. Since $A \equiv B$, there exists a bijection $f : A \to B$. Since $B \equiv C$, there exists a bijection $g : B \to C$. Then, by Exercise 1.3.6, we have that $g \circ f : A \to C$ is a bijection and so $A \equiv C$. □

Definition 1.4.3. We say a set A is **finite (or countably finite)** if, for some $n \in \mathbb{Z}_+$, A has the same cardinality as $\mathbb{Z}_n$ and we say the cardinality of A is n, or write $card(A) = |A| = n$. We say a set A is **countably infinite** if it has the same cardinality as $\mathbb{N}$ and we say the cardinality of A is $\aleph_0$, or write $card(A) = |A| = \aleph_0$. If A is finite or countably infinite then we say A is **countable**. If A is not countable, then we say A is **uncountable**. If A has the same cardinality as $\mathbb{R}$, then we say the cardinality of A is the **continuum** and write $card(A) = |A| = \mathfrak{c}$.

To avoid spending too much time on cardinality, we will assume the reader has already examined the following statements and theorems in another course so we will discuss them without proof. Obviously, if the reader is interested in seeing the proofs of any of these theorems then the instructor can direct them to an appropriate source.

The cardinalities of our most common sets are:

$$|\mathbb{Z}_n| = n$$

$$|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| = \aleph_0$$

$$|\mathbb{R}| = |\mathbb{C}| = \mathfrak{c}$$

From the above notation and terminology, one would assume that $\mathbb{R}$ and $\mathbb{C}$ are uncountable (otherwise we would have no use for the symbol $\mathfrak{c}$) and indeed, this is the case.

The fact that the cardinality relation forms an equivalence relation gives a very common strategy for showing a set has a certain cardinality, say γ. That is, if we want to show a set A has cardinality γ, and we already know a set B has cardinality γ, then it is enough to show that there exists a bijection $f : A \to B$ or a bijection $g : B \to A$ (there is a bijection $f : A \to B$ if and only if there is a bijection $g : B \to A$, namely $g = f^{-1}$).

So, for example, if we want to show a set A is countably infinite, we have many options. We could show that there is a bijection $f : A \to \mathbb{N}$, a bijection $g : \mathbb{Q} \to A$, or a bijection $h : A \to \mathbb{Z}$, just to list a few possibilities.

Another useful theorem is the following:

Theorem 1.4.4. *Let A and B be countable sets. Then $A \cup B$, $A \cap B$, and $A \times B$ are countable.*

Note that this theorem then extends to the following corollary:

Corollary 1.4.5. *Let $A_1, A_2, \ldots, A_n$ be countable sets. Then*

$$\bigcup_{i \in Z_n} A_i \quad , \quad \bigcap_{i \in Z_n} A_i \quad , \quad A_1 \times A_2 \times \cdots \times A_n$$

are all countable sets.

We do have to be careful with the above theorem and corollary when we are making a distinction between countably finite and countably infinite. For example, it is possible that A and B are countably infinite but $A \cap B$ is countably finite. Also, if A is countably finite and B is countably infinite then $A \cup B$ and $A \times B$ would be countably infinite.

Another useful theorem is the following:

Theorem 1.4.6. *Let A and B be sets where $A \subseteq B$. If B is countable, then A is countable. On the otherhand, if A is uncountable then B is uncountable.*

There is also a way to order our cardinalities. If A and B are sets and there exists an injection $f : A \to B$ but no surjection $g : A \to B$ then we write $|A| < |B|$.

With this notation, we then have

$$0 < 1 < 2 < \cdots < \aleph_0 < \mathfrak{c}$$

if we think of $1, 2, \ldots$ as cardinalities rather than numbers.

This string of inequalities might raise several questions to an inquisitive reader and, indeed, sparked a lot of interest among mathematicians back in the day. The first line of questioning might be, "Does there exist a set A such that $\mathfrak{c} < |A|$? If such a set A exists, does there exist a set B so that $|A| < |B|$?" A second question might be, "Can we find a set A such that $\aleph_0 < |A| < \mathfrak{c}$?"

To answer the first line of questioning, we have the following theorem due to Cantor.

Theorem 1.4.7. (Cantor's Theorem) *Let A be a set. Then $|A| < |\mathcal{P}(A)|$.*

Proof. following exercise □

Exercise 1.4.8. Prove Cantor's Theorem. *Hint: Clearly, we can define an injection from a set A to $\mathcal{P}(A)$ (just map every element $x \in A$ to $\{x\}$). So, we just need to show that there can be no surjection $f : A \to \mathcal{P}(A)$. Do this by contradiction. Suppose we have such a surjection f. Define $X = \{a \in A : a \notin f(a)\}$. Since $X \in \mathcal{P}(A)$ and f is a surjection, there exists $a_0 \in A$ such that $f(a) = X$. Now, ask the question, "is $a_0 \in f(a_0)$?".*

Hence, from Cantor's Theorem, we can see that

$$\mathfrak{c} = |\mathbb{R}| < |\mathcal{P}(\mathbb{R})| < |\mathcal{P}(\mathcal{P}(\mathbb{R}))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{R})))| < \cdots$$

and so we have our answer to the first line of questioning. Recall from combinatorics the fact that if A is a set and $|A| = n$, then $|\mathcal{P}(A)| = 2^n$. For this reason, you will often see, even when A is countably infinite or uncountable, the cardinal number $2^{|A|}$, by which we mean $2^{|A|} = |\mathcal{P}(A)|$. For example, $2^{\aleph_0} = |\mathcal{P}(\mathbb{N})| = |\mathcal{P}(A)|$ for any countably infinite set A.

Perhaps Cantor's Theorem also gives us an answer to our second question. That is, we know from Cantor's Theorem that $\aleph_0 < 2^{\aleph_0}$ so we will have our answer if $2^{\aleph_0} < \mathfrak{c}$. Unfortunately, this is not the case as the next theorem shows.

Theorem 1.4.9.

$$2^{\aleph_0} = \mathfrak{c}$$

The second question, "Does there exist a set A such that $\aleph_0 < |A| < \mathfrak{c}$?" remained an open question, called The Continuum Hypothesis, for quite a long time. To be precise, the Continuum Hypothesis is the statement, "There is no set A such that $\aleph_0 < |A| < \mathfrak{c}$." It was proposed by Cantor in the 1870s and it is worded this way because Cantor believed that there was no cardinality strictly between $\aleph_0$ and $\mathfrak{c}$.

So, what is the answer? It's complicated. To understand, we first need to discuss **Gödel's Incompleteness Theorem**. In 1931, Kurt Gödel proved that, essentially, given any "appropriate" axiomatic system, there will always be mathematical statements which can't be proven or disproven. While the theorem is remarkable, it was definitely a cause for concern for mathematicians at the time. To be clear, it says that no matter what list of axioms you start with (as long as it's an appropriate one, i.e., not self-contradictory, sufficiently large enough to include basic arithmetic, etc.) there will always be questions we can't answer or statements whose truth value we cannot verify. We say such statements are "independent of the axioms." So, when confronted with a statement and deciding on its truth value, we don't just have the two possible answers of "true" and "false" but, rather, the three possible answers of "true," "false," or "independent of the axioms."

At this point, the reader probably sees where this is going. It turns out, through a proof given by Gödel in the 1940s and another by Cohen in the 1960s, that the Continuum Hypothesis is independent of the ZFC axioms. That is, starting with the ZFC axioms of set theory, it is impossible to prove or disprove the Continuum Hypothesis. When the Incompleteness Theorem was first published, the only examples of statements which were indepedent of a given axiomatic system, were not particularly important ones and often even rather ridiculous. At the time, this gave mathematicians at least some comfort. The fact that the Continuum Hypothesis turned out to be one of these independent statements was the first example of a rather important statement being independent of the axioms.

1.5 Axiom of Choice

It is now time to exam the "C" in "ZFC" which was mentioned in the beginning of this chapter.

Definition 1.5.1. (Axiom of Choice) Let X be a set whose elements are all nonempty sets. Then there exists a function f whose domain is X such that $f(A) \in A$ for all $A \in X$.

One thing that is nice about having statements which are independent of a set of axioms is that we can then include those statements as a new axiom without fear of it causing problems, at least as a logical system. Whether or not it is appropriate for the statement to be an axiom is usually cause for much debate. The Axiom of Choice turns out to be independent of the ZF Axioms and so we can include it as a new axiom, and, when we do, we end up with the ZFC Axioms.

There was a time when mathematicians debated on whether or not it was appropriate to include the Axiom of Choice as an axiom for set theory. Indeed, some pretty strange and counterintuitive statements can be proven using the Axiom of Choice (for example, the Banach-Tarski Paradox, which says that it is possible to decompose the unit ball in $\mathbb{R}^3$ into finitely many pieces and then, using only rotations and translations, reassemble the pieces into two identical copies of the unit ball, each having the same volume as the original). Nevertheless, today it is universally accepted as an appropriate axiom and we can use it freely.

The way the Axiom of Choice is worded is not usually how it is used. Suppose we have an index set I and, for each $i \in I$, there is a nonempty set A_i. The Axiom of Choice allows us to then say things like, "for each $i \in I$, since A_i is nonempty, let $a_i \in A_i$." It doesn't seem so controversial now, does it? If I is finite, then we don't actually need the Axiom of Choice. Or, if we specified how exactly one should pick the $a_i \in A_i$, then the Axiom of Choice is not needed either. The Axiom of Choice is used when we have an arbitrary collection of nonempty sets and we want to pick an element out of each of these sets without telling the reader *how* to pick each element.

There are many statements which are equivalent to the Axiom of Choice. One of which is called Zorn's Lemma. Since we freely use the Axiom of Choice, we will also freely use Zorn's Lemma.

Theorem 1.5.2. (Zorn's Lemma) *Let P be a nonempty partially ordered set. If every chain in P has an upper bound in P, then P has at least one maximal element.*

Admittedly, it is not obvious that Zorn's Lemma is equivalent to the Axiom of Choice. Again, we will omit the proof to avoid spending too much time on preliminaries. The interested reader would certainly have no trouble finding a proof in the literature.

Exercise 1.5.3. Let A and B be sets and let $f : A \to B$. Let $C \subseteq A$. We define $f|_C : C \to B$ by $f|_C(c) = f(c)$ for all $c \in C$ and call $f|_C$ the **restriction of f to C**

. Suppose $f : A \to B$ is surjective. Prove there exists $C \subseteq A$ such that $f|_C : C \to B$ is a bijection. *Hint: Use the Axiom of Choice.*

1.6 Products of Sets Revisited

We have already seen how to define the product of finitely many sets. In order to take products of infinitely many sets (countable or uncountable), it is worth revisiting the product of a finite number of sets to develop a different view.

As we saw in an earlier section, given $n \in \mathbb{Z}_+$, and sets $A_1, A_2, \ldots, A_n$, the product $A_1 \times A_2 \times \cdots \times A_n$ is the set of all n-tuples $(a_1, a_2, \ldots, a_n)$. A notation we will be making use of is

$$\prod_{i=1}^{n} A_i = A_1 \times A_2 \times \cdots \times A_n.$$

Now, another way to view the elements of the above set is to associate the n-tuple $(a_1, a_2, \ldots, a_n)$ with the function $f : \mathbb{Z}_n \to \cup_{i=1}^{n} A_i$, where $f(i) = a_i$ for all $i = 1, 2, \ldots, n$. Note that we have $f(i) \in A_i$ for all $i = 1, 2, \ldots, n$. Thus, we can think of $\prod_{i=1}^{n} A_i$ as the set of all functions $f : \mathbb{Z}_n \to \cup_{i=1}^{n} A_i$ such that $f(i) \in A_i$ for all $i = 1, 2, \ldots, n$.

If instead, we have a countable collection of sets, say $A_1, A_2, \ldots$, then one way to view the set

$$\prod_{i=1}^{\infty} A_i = A_1 \times A_2 \times \cdots$$

is that it is the set of all sequences $(a_1, a_2, \ldots)$, where $a_i \in A_i$ for all $i \in \mathbb{Z}_+$. Another way to view this set is that it is the set of all functions $f : \mathbb{Z}_+ \to \cup_{i=1}^{\infty} A_i$ where $f(i) \in A_i$ for all $i \in \mathbb{Z}_+$.

Now, suppose we have an index set I and, for each $i \in I$, a set A_i. If I is finite or countable then we can interpret $\prod_{i \in I} A_i$ as we did in the above two situations. But what if I is uncountable? How do we picture an ordered tuple of uncountably many elements? To make matters worse, what if our index set I isn't a totally ordered set? Then what would the order of our coordinates even be? For these reasons, it is better to have the function interpretation of the product of these sets and think of $\prod_{i\in I} A_i$ as the set of all functions $f : I \to \cup_{i\in I} A_i$ such that $f(i) \in A_i$.

Example 1.6.1. (i) Suppose $I = \{1, 2\}$ and $A_1 = A_2 = \mathbb{R}$. Then $\prod_{i\in I} A_i = \mathbb{R}^2$ or, if we'd rather, $\prod_{i\in I} A_i$ is the set of all functions $f : \{1, 2\} \to \mathbb{R}$.

(ii) Suppose $I = \mathbb{Z}_+$ and $A_i = \mathbb{R}$ for all $i \in \mathbb{Z}_+$. Then $\prod_{i\in I} A_i$ is the set of all real-valued sequences $(x_i)_{i=1}^{\infty}$ that we study in our Calculus classes. If, in the first example, we think of 2 as a cardinal number rather than a positive integer, and think about the equation $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$, then it is natural in this example to write $\prod_{i\in\mathbb{Z}_+} \mathbb{R} = \mathbb{R}^{\aleph_0}$. It is more common though, by an abuse of notation, to instead see $\prod_{i\in\mathbb{Z}_+} \mathbb{R} = \mathbb{R}^{\mathbb{Z}_+}$ because it has the advantage of specifying the index set specifically. One advantage to this, is when viewing the elements of the product as functions, we have told the readers which countable set, specifically $\mathbb{Z}_+$, we are using as the domain of our functions.

(iii) Suppose $I = \mathbb{R}$ and $A_i = \mathbb{R}$ for all $i \in I$. Then $\prod_{i\in I} A_i = \mathbb{R}^{\mathbb{R}}$ is the set of all functions $f : \mathbb{R} \to \mathbb{R}$. Similarly, $\mathbb{R}^{\mathbb{C}}$ is the set of all functions $f : \mathbb{C} \to \mathbb{R}$ while $\mathbb{C}^{\mathbb{R}}$ is the set of all functions $f : \mathbb{R} \to \mathbb{C}$. Thus, for example, if we define $f : \mathbb{C} \to \mathbb{R}$ by $f(z) = |z|$ (where $|z|$ is the modulus of z), then $f \in \mathbb{R}^{\mathbb{C}}$. Also, if $g : \mathbb{R} \to \mathbb{C}$ is defined by $g(x) = e^{ix}$, then $g \in \mathbb{C}^{\mathbb{R}}$.

Example 1.6.2. Some examples of subsets of $\mathbb{R}^{\mathbb{Z}_+}$, $\mathbb{R}^{\mathbb{R}}$, and $\mathbb{R}^{[a,b]}$, where $[a, b]$ is a closed interval, we will be discussing later in the notes are given here.

(i) Let c_{00} be the set of all real-valued sequences with finitely many nonzero coordinates. Then $c_{00} \subseteq \mathbb{R}^{\mathbb{Z}_+}$.

(ii) Let c_0 be the set of all real-valued sequences which converge to zero. Then $c_0 \subseteq \mathbb{R}^{\mathbb{Z}_+}$.

(iii) Let c be the set of all real-valued sequences which converge. Then $c \subseteq \mathbb{R}^{\mathbb{Z}_+}$.

(iv) Let A be a set. Denote $C(A)$ to be the set of all continuous functions $f : A \to \mathbb{R}$. Then $C(\mathbb{R}) \subseteq \mathbb{R}^{\mathbb{R}}$ and $C([a,b]) \subseteq \mathbb{R}^{[a,b]}$.

Note that we have $c_{00} \subset c_0 \subset c \subset \mathbb{R}^{\mathbb{Z}_+}$.

Exercise 1.6.3. Let I be an index set and suppose A_i is a nonempty set, for all $i \in I$. Prove that $\prod_{i \in I} A_i$ is nonempty. *Hint: Use the Axiom of Choice.*

Index