Pettis Integration with Applications to Generators of Quantum Markov Semigroups

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Abstract

The mathematical formulation of quantum statistical mechanics relies heavily on areas of functional analysis, particularly operator theory. In this thesis, we will look at several topics of operator theory that are useful in the study of quantum statistical mechanics. We will then apply this knowledge to better understand quantum Markov semigroups, a quantum analogue of classical Markov semigroups, which are used to model irreversible open quantum systems.

The first topic of this thesis is integration of Banach space valued functions. For our purposes we are interested in a weak type of integration, namely Pettis integration. In our study of quantum Markov semigroups, classical Pettis integration is of much use but it was found that a more general concept of a Pettis integral was needed, so some time is spent defining this generalization and developing theorems which give sufficient conditions for a Banach space valued function to be integrable.

The form of the generator of a quantum Markov semigroup has been discussed quite extensively since the 1970s, when a form was given for uniformly continuous quantum Markov semigroups. Unfortunately many important examples of quantum Markov semigroups are not uniformly continuous. In this thesis it is shown that without the assumption of uniform continuity, the generator still has a similar form. Several examples are then examined in order to clarify the findings. Lastly, the concept of Pettis integration is used in order to show that the generator of a quantum Markov semigroup is closed in a suitable sense. In doing so, added assumptions are needed with regard to the elements of the semigroup. Several examples of quantum Markov semigroups which satisfy these assumptions are presented.

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Chapter 1

Introduction and mathematical background

1.1 Introduction and overview

In this section a motivation and overview of the results is presented, whereas precise definitions appear in section 1.3. In the early seventies, R.S. Ingarden and A. Kossakowski [25, 28] postulated that the time evolution of a statistically open system, in the Schrodinger picture, be given by a one-parameter semigroup of linear operators acting on the trace-class operators of a separable Hilbert space $\mathcal{H}$ satisfying certain conditions. In the Heisenberg picture the situation translates to a one-parameter semigroup $(T_t)_{t\geq 0}$ acting on $\mathcal{B}(\mathcal{H})$ (the bounded operators on a Hilbert space $\mathcal{H}$) where each T_t is positive and σ-weakly continuous, satisfying $T_t(1) = 1$ for all $t \geq 0$, and where the map $t \mapsto T_t A$ is σ-weakly continuous for each $A \in \mathcal{B}(\mathcal{H})$.

In 1976, G. Lindblad [31] added to the formulation the condition that each T_t be completely positive rather than simply positive, a condition which he justified physically. Results of Stinespring [39, Theorem 4] and Arveson [1, Proposition 1.2.2] further justify this condition by proving that if an operator has a commutative domain or target space then positivity and complete positivity are equivalent. Further, under the assumption that the map $t \mapsto T_t$ is uniformly continuous, the semigroup is called a uniformly continuous QMS, the generator L of the semigroup is bounded, and Lindblad was able to write L in the form $L(A) = \phi(A) + G^*A + AG$ where ϕ is completely positive and $G \in \mathcal{B}(\mathcal{H})$. Using an earlier theorem of Stinespring [39] the map ϕ can be written in the form $\phi(A) = V^*\pi(A)V$ where $V : \mathcal{H} \to \mathcal{K}$ for some

Hilbert space $\mathcal{K}$ and $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is a normal representation. Further, a theorem due to Kraus [29] implies that π can be written in the form $\pi(A) = \sum_{n=1}^{\infty} W_n^* A W_n$ where $W_n : \mathcal{K} \to \mathcal{H}$ is a bounded linear operator. When Stinespring's and Kraus' results are combined one is able to write ϕ in the form $\phi(A) = \sum_{n=1}^{\infty} V_n^* A V_n$ where $V_n \in \mathcal{B}(\mathcal{H})$. Lindblad's original result was for QMSs on a hyperfinite factor $\mathfrak{A}$ of $\mathcal{B}(\mathcal{H})$ (which includes the case $\mathfrak{A} = \mathcal{B}(\mathcal{H})$, see Topping [41]). A similar result to Lindblad's was given in that same year by Gorini, Kossakowski, and Sudarshan [20] for QMSs on finite dimensional Hilbert spaces and three years later Christensen and Evans [10] proved it for uniformly continuous QMSs on arbitrary von Neumann algebras. A nice exposition of these results was written by Fagnola [18]. Another name for QMSs that appears in the literature is CP_0-semigroups; for example see Arveson [3]. An important subclass of QMSs that has also attracted a lot of attention is the class of E_0-semigroups which was introduced by Powers [35].

While this thesis is centered around the question, "Given a QMS, what is the form of its generator and does it satisfy the Lindblad equation?," the converse question, "Given an operator on $\mathcal{B}(\mathcal{H})$ which has the Lindblad form, does there exist a QMS whose generator is the given operator?," has attracted a lot of attention in the literature. The study of this converse question was originated by Davies [11].

In this thesis, analogous results to Lindblad and Stinespring are proved, and some progress is made towards extending the results of Kraus for the generator of a QMS acting on $\mathcal{B}(\mathcal{H})$ when the condition that the semigroup is uniformly continuous is no longer assumed. In this case, the generator L is no longer bounded, and so, inevitably, much discussion on domains of operators and the density of such domains is required. Because of such difficulties, the notion of U-completely positive maps (for a linear subspace U of $\mathcal{H}$) is introduced which is analogous to the notion of completely positive maps but is better suited for unbounded operators (see Definition 2.2.2). It is then shown (see Theorem 2.2.3) that if L denotes the generator of a QMS on $\mathcal{B}(\mathcal{H})$, there exists a subspace W of $\mathcal{H}$, a linear operator $K : W \to \mathcal{H}$, and a W-completely positive map $\phi : D(L) \to S(W)$ (where $D(L)$ denotes the domain of L and $S(W)$ denotes the set of sesquilinear forms on $W \times W$) such that

$$\langle u, L(A)v\rangle = \phi(A)(u, v) + \langle Ku, Av\rangle + \langle u, AKv\rangle$$

for all $A \in D(L)$ and all $u, v \in W$. Unfortunately this result does not give a lot of information about the subspace W or the operator K. On the other hand, if one restricts A to belong to the domain algebra $\mathcal{A}$ of L, which is the largest *-subalgebra of the domain of L and was studied by Arveson [2], then it is proved (see Theorem 2.2.6) that there exists an explicit subspace U of $\mathcal{H}$ and a linear operator $G : U \to \mathcal{H}$ having an explicit formula and a U-completely positive map $\phi : \mathcal{A} \to S(U)$ such that

$$\langle u, L(A)v\rangle = \phi(A)(u, v) + \langle u, GAv\rangle + \langle GA^*u, v\rangle$$

for all $A \in \mathcal{A}$ and for all $u, v \in U$ where $\phi : \mathcal{A} \to S(U)$ is U-completely positive.

With regard to Stinespring, it is shown (see Theorem 2.2.8) that there exists a Hilbert space $\mathcal{K}$, a linear map $V : \mathcal{H} \to \mathcal{K}$, and a unital *-representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ so that $\phi(A)(u, w) = \langle Vu, \pi(A)Vw\rangle$ for all $u, w \in U$. By putting these results together the generator of the QMS is written in the form

$$\langle x, L(A)y\rangle = \langle Vx, \pi(A)Vy\rangle + \langle x, GAy\rangle + \langle GA^*x, y\rangle, \quad \text{for all } x, y \in U \tag{1.1}$$

where

$$Gx = L(|x\rangle\langle e|)e - \frac{1}{2}\langle e, L(|e\rangle\langle e|)e\rangle x, \quad \text{for all } x \in U.$$

Theorems 2.2.6 and 2.2.8 are summarized in Corollary 2.2.9 which is the main result of the thesis. Similar results have been given in the literature. In 1979, Davies [12] showed that for the generator of a strongly continuous semigroup of completely positive contractions on the space of trace class operators (the form of the predual semigroup of a QMS) to have a form similar to Lindblad's, it is enough for $(T_t)_{t\geq 0}$ to possess a pure stationary state, that is, an $x \in \mathcal{H}$ such that $T_t(|x\rangle\langle x|) = |x\rangle\langle x|$, for all $t \geq 0$, and for the dual semigroup to map compact operators to compact operators or for the dual semigroup to satisfy an extension property which is given in Davies [12, pg. 429]. The results of this thesis differ from this result of Davies' since it is neither assumed that the QMS is strongly continuous nor that a pure stationary state exists, but rather it is assumed that a rank-one operator is contained in the domain of the generator. Another result in this direction was given by Holevo [23] in 1993 which, though similar to the main result of this thesis, contains significant differences. First, Holevo assumes there exists a dense subspace $\mathcal{D}$ of $\mathcal{H}$ such that for any $x, y \in D$

$$\lim_{t \to 0} \frac{1}{t} \langle x, T_t(A)y \rangle$$

exists for all $A \in \mathcal{B}(\mathcal{H})$. On the other hand, for the results of this thesis it is not assumed that this limit exists for all $A \in \mathcal{B}(\mathcal{H})$. Instead Arveson's domain algebra is used. Further, in the main result of this thesis, the equation of the sesquilinear forms satisfied by the generator of a QMS is valid on a subspace U of $\mathcal{H}$ just, as Holevo's sesquilinear form is only valid on its domain $\mathcal{D}$. Moreover in the main result of the thesis the set U is very specific and Lindblad's form is obtained without needing to assume that U is dense. In fact, in Section 2.4 an example is provided where U is not dense and the main result of the thesis is still valid. In Section 2.3 partial results are given similar to those given by Kraus but fall slightly short and discuss a possible way forward (see Proposition 2.3.6 and the discussion that follows it). Under the assumption

$$\text{there exists } C > 0 \text{ such that } \|L(|x\rangle\langle Th|)h\| \leq C\|x\| \text{ for all } x \in U \tag{1.2}$$

(where T is a positive finite rank operator in the domain of L and h is an appropriate vector) it is shown that the unital *-representation $\pi : \mathcal{A} \to \mathcal{K}$ which appears in Lindblad's form is σ-weakly - σ-weakly closable. Unfortunately this assumption is rather strong as it implies V and G are bounded and L is bounded on its domain which are properties not often valid in examples. If instead the following weaker assumption is adopted,

$$\forall x \in U,\ \exists C_x > 0 \text{ such that } |\langle x, L(|y\rangle\langle e|)e\rangle| \leq C_x \|y\| \quad \text{for all } y \in U \tag{1.3}$$

(where $|e\rangle\langle e| \in D(L)$) then we no longer have that the operators G and V are bounded. Further under the assumption (1.3), it can be easily proved that

(i) U is a subset of the domain $Dom(G^*)$ of G^*, and

(ii) for every $y \in U$ and $A \in \mathcal{A}$ we have that $\pi(A)Vy \in Dom(V^*)$.

Statements (i) and (ii) imply that Equation (1.1) can be equivalently rewritten as

$$\langle x, L(A)y\rangle_{\mathcal{H}} = \langle x, \left(V^*\pi(A)V + GA + AG^*\right) y\rangle_{\mathcal{H}} \quad \text{for all } A \in \mathcal{A} \text{ and } x, y \in U. \tag{1.4}$$

If it is further assumed that

$$\text{the linear subspace } U \text{ is dense in } \mathcal{H} \tag{1.5}$$

then Equation (1.4) can be equivalently written as

$$L(A)y = \left(V^*\pi(A)V + GA + AG^*\right) y \text{ for all } A \in \mathcal{A} \text{ and } y \in U. \tag{1.6}$$

Thus, under assumptions (1.3) and (1.5) the equality (1.1) of sesquilinear forms becomes equality (1.6) of operators.

The operators $V : U \to \mathcal{H}$ and $G : U \to \mathcal{K}$ and their adjoints that appear in (1.6) are not necessarily bounded. Let $\mathcal{L}(U)$ denote the set of all (not necessarily bounded) linear operators on $\mathcal{H}$ whose domains contain the linear subspace U of $\mathcal{H}$. In order to prove that the *-representation π which appears in (1.6) is continuous with respect to some topologies one needs to show that the map $\varphi : \mathcal{A} \to \mathcal{L}(U)$ defined by

$$\varphi(A) = V^*AV, \quad A \in \mathcal{A} \tag{1.7}$$

is closed with respect to appropriate topologies. Given a QMS $(T_t)_{t\geq 0}$, the topologies with respect to which each map T_t is continuous, are the same as the topologies used for the closedness of the map φ defined in (1.7). Even though for a general QMS it is assumed that every T_t is σ-weakly-σ-weakly continuous, the σ-weak topology cannot be naturally defined in the set $\mathcal{L}(U)$ of generally unbounded operators (which contains the range of the map φ). On the other hand, a natural topology on the space $\mathcal{L}(U)$ of linear operators whose domain contains the dense linear subspace U, is the **weak operator topology on** U, which will be denoted by $WOT(U)$. The $WOT(U)$ topology is the locally convex topology generated by the family of seminorms $(p_{x,y})_{x,y\in U}$ where, for $x, y \in U$ and $T \in \mathcal{L}(U)$, we define $p_{x,y}(T) = |\langle x, Ty\rangle|$.

For these reasons Chapter 3 of this thesis is devoted to the study of semigroups such that each member of the semigroup is continuous with respect to various weak topologies, and it is proved that the generator of the semigroup is closed with respect to these topologies. The main results in this direction are Theorems 3.1.9 and 3.1.11 and Corollary 3.2.1 which give sufficient conditions for the generator of a semigroup to be closed with respect to various weak topologies. The main tools for proving these results are Theorems 1.2.7 and 1.2.9 which provide sufficient conditions for a Banach space valued function to be Pettis integrable.

1.2 Sufficient conditions for Pettis integrability

In this section we first recall Pettis integrability of Banach space valued functions as in [14]. The main results of the section are Theorems 1.2.7 and 1.2.9 which give sufficient conditions for a function to be Pettis integrable. These theorems are similar to results in [30].

Definition 1.2.1. *Let (Ω, Σ, μ) be a measure space. Let X be a normed space and let F be a linear subspace of X^*. If $f : \Omega \to X$ is such that $\eta \circ f$ is Σ-measurable for all $\eta \in F$ then we say that f is **F-measurable**. Further, if f is F-measurable and $\eta \circ f \in L_1(\Omega)$ for all $\eta \in F$ then we say f is **F-Dunford integrable**.*

Proposition 1.2.2. *Let (Ω, Σ, μ) be a measure space. Let X be a Banach space and F be a norm closed subspace of X^*. If $f : \Omega \to X$ is F-Dunford integrable then for any $E \in \Sigma$, there exists $B_E \in F^*$ such that*

$$B_E(\eta) = \int_E \eta \circ f(t) d\mu(t), \quad \textit{for all } \eta \in F.$$

Proof. Let $E \in \Sigma$ and define $T : F \to L_1(\Omega)$ by $T(\eta) = (\eta \circ f)\chi_E$ for all $\eta \in F$. We first show that T is closed so let a sequence $(\eta_n)_{n\in\mathbb{N}} \subseteq F$ such that $\eta_n \to \eta$ for some $\eta \in F$ (with respect to the norm of F), and $T(\eta_n) \to g$ in $L_1(\Omega)$. Then, there exists a subsequence $(\eta_{n_k})_{k\in\mathbb{N}}$ such that $T(\eta_{n_k}) \to g$ almost everywhere, hence $(\eta_{n_k} \circ f)(t)\chi_E(t) \to g(t)$ for almost every $t \in \Omega$. On the other hand, since $\eta_n \to \eta$ in norm we have that $(\eta_n \circ f)(t)\chi_E(t) \to (\eta \circ f)(t)\chi_E(t)$ for all $t \in \Omega$. Thus we must have that $T(\eta) = g$ and so T is closed. Since F is a Banach space, by the closed graph theorem, T is continuous and so

$$\int_E |\eta \circ f(t)| d\mu(t) = \|T(\eta)\|_1 \leq \|T\|\|\eta\|.$$

Hence the linear map $B_E : F \to \mathbb{C}$ defined by

$$B_E(\eta) = \int_E \eta \circ f(t) d\mu(t)$$

is continuous on F and therefore $B_E \in F^*$.

□

In the situation described in Proposition 1.2.2 we write

$$B_E = (D) - \int_E f(t) d\mu(t).$$

We would like a similar result to the above in the case when F is a linear subspace of X^* which is not necessarily closed. This is achieved by replacing the assumption "f is F-Dunford integrable" by the stronger assumption "$\|f(t)\| \leq g(t)$ for μ-almost all $t \in \Omega$ for some $g \in L_1(\Omega)$" as in the following:

Proposition 1.2.3. *Let (Ω, Σ, μ) be a measure space. Let X be a Banach space and F be a linear subspace of X^*. Further, suppose $f : \Omega \to X$ is Σ-measurable. If there exists $g \in L_1(\Omega)$ such that $\|f(t)\| \leq g(t)$ for μ-almost all $t \in \Omega$ then f is F-Dunford integrable and for any $E \in \Sigma$, there exists $B_E \in F^*$ such that*

$$B_E(\eta) = \int_E \eta \circ f(t) d\mu(t), \quad \textit{for all } \eta \in F.$$

Proof. Clearly $\eta \circ f \in L_1(\Omega)$ for all $\eta \in F$ so f is F-Dunford integrable. Let $E \in \Sigma$. Define $B_E : F \to \mathbb{C}$ by

$$B_E(\eta) = \int_E \eta \circ f(t) d\mu(t), \quad \text{for all } \eta \in F.$$

Then, for any $\eta \in F$, we have

$$|B_E(\eta)| \leq \int_E |\eta \circ f(t)| d\mu(t) \leq \|\eta\| \int_E \|f(t)\| d\mu(t) \leq \|\eta\| \int_E g(t) d\mu(t) \leq \|\eta\| \|g\|_1$$

and so B_E is bounded and hence $B_E \in F^*$ which completes the proof. □

Definition 1.2.4. *Let (Ω, Σ, μ) be a measure space, X a normed space, and F a subspace of X^*. If $f : \Omega \to X$ is F-Dunford integrable and*

$$(D) - \int_E f(t) d\mu(t) \in X$$

*for all $E \in \Sigma$ then f is said to be an **F-Pettis integrable function with X-valued integrals** and we write*

$$(D) - \int_E f(t) d\mu(t) = (P) - \int_E f(t) d\mu(t).$$

When the measure μ is understood then we do not have to specify which measure is considered for Dunford and Pettis integrability. Otherwise, we will specify the measure with respect to which the Dunford or Pettis integrability is concerned.

At first it may seem redundant that for an F-Pettis integrable function f we specify that it has X-valued integrals. This detail is important when we allow X to be non-complete as it can be seen in Theorems 1.2.7 and 3.1.11 and Corollary 3.2.1. If X is complete then we do not insist on mentioning the fact that the integrals are in X (since there is no superspace of X to cause confusion about the containment of the integrals). In this case we simply say "f is an **F-Pettis integrable function**".

It is worth noting that if a Banach space X has a predual, say X_*, and $F = X_*$ then F-Dunford integrability automatically implies F-Pettis integrability because, by Proposition 1.2.2, the F-Dunford integral $B_E \in F^* = X$. One way to determine if a Banach space X has a predual is the following result.

Proposition 1.2.5. *Let X be a Banach space and let F be a linear subspace of X^*. If the $\sigma(F, X)$ closure of every convex subset of F is equal to its norm closure then $F^* = X$.*

Proof. Let $B \in F^*$. We want to show that B is $\sigma(F, X)$ continuous. Suppose this is not the case. Then there exists a net $(\eta_i)_{i\in I} \subseteq F$ and $\eta \in F$ such that $\eta_i \to \eta$ in the $\sigma(F, X)$ topology but $B(\eta_i)$ does not converge to $B(\eta)$. Thus there exists $\epsilon > 0$ so that for every $\alpha \in I$ there exists $\beta \geq \alpha$ such that $B(\eta_\beta) \notin O$ where

$$O = (\Re(B(\eta)) - \epsilon, \Re(B(\eta)) + \epsilon) \times (\Im(B(\eta)) - \epsilon, \Im(B(\eta)) + \epsilon) .$$

So we can define a subnet $(B(\eta_j))_{j\in J}$ such that $B(\eta_j) \notin O$ for all $j \in J$. Since,

$$\begin{aligned} O^c = \{z : \Re z \geq \Re(B(\eta)) + \epsilon\} &\cup \{z : \Re z \leq \Re(B(\eta)) - \epsilon\} \\ \cup \{z : \Im z \geq \Im(B(\eta)) + \epsilon\} &\cup \{z : \Im z \leq \Im(B(\eta)) - \epsilon\} \end{aligned}$$

and $\{j \in J : B(\eta_j) \in O^c\}$ is a final set which can be written as a union of four sets, at least one of these four sets is a final set. Thus there exists a closed "half" plane H of $\mathbb{C}$ such that $J_0 = \{j \in J : B(\eta_j) \in H\}$ is a final set. Now, $\eta_j \underset{J_0}{\to} \eta$ in the $\sigma(F, X)$ topology so $\eta \in \overline{co\{\eta_j : j \in J_0\}}^{\sigma(F,X)}$. Then, by assumption $\eta \in \overline{co\{\eta_j : j \in J_0\}}^{\sigma(F,X)} = \overline{co\{\eta_j : j \in J_0\}}^{\|\cdot\|}$. Hence, there exists a sequence $(\zeta_k)_{k\geq 1} \subseteq$

$co\{\eta_j : j \in J_0\}$ such that $\zeta_k \to \eta$ in norm. Since $\zeta_k \in co\{\eta_j : j \in J_0\}$ we have that $\zeta_k = \sum_{j\in J_k} \alpha_{k,j}\eta_j$ for some $J_k \subseteq J_0$ and $\alpha_{k,j} \geq 0$ where $\sum_{j\in J_k} \alpha_{k,j} = 1$. Then

$$B(\zeta_k) = \sum_{j\in J_k} \alpha_{k,j} B(\eta_j)$$

and $B(\eta_j) \in H$ for all $j \in J_k$ so $\sum_{j\in J_k} \alpha_{k,j} B(\eta_j) \in H$ as a convex combination of the elements $B(\eta_j)$'s of the convex set H. So $B(\zeta_k) \in H$ for all $k \in \mathbb{N}$ and $B(\zeta_k) \to B(\eta)$, hence $B(\eta) \in H$ since H is closed which is a contradiction. So we must have that B is $\sigma(F, X)$ continuous and therefore $F^* = X$. □

Definition 1.2.6. *Let X be a normed space and let F be a subspace of X^* which separates points in X (thus the $\sigma(X, F)$ topology on X is Hausdorff). We say the pair (X, F) satisfies the* ***Krein-Smulian Property*** *if the $\sigma(X, F)$ closure of the convex hull of any norm-bounded $\sigma(X, F)$ compact subset of X is $\sigma(X, F)$ compact.*

The previous definition is motivated by the classical Krein-Smulian Theorem which states that for any Banach space X, the pair (X, X^*) satisfies the Krein-Smulian property. The next result strengthens Proposition 2.5.18 of [4].

Theorem 1.2.7. *Let X be a normed space and let F be a norming linear subspace of X^* such that (X, F) satisfies the Krein-Smulian property. Suppose (Ω, τ) is a topological space, Σ is the Borel σ-algebra generated by τ and $\mu : \Sigma \to [0, \infty]$ is a σ-compact measure. Let $\overline{X}$ be the completion of X, (hence $\overline{X}$ is a Banach space), and $f : \Omega \to X \subseteq \overline{X}$ be a function such that $\eta \circ f$ is continuous for all $\eta \in F$. If there exists $g \in L_1(\Omega, \Sigma, \mu)$, which is bounded on compact sets, and satisfies $\|f(t)\| \leq g(t)$ for μ-almost all $t \in \Omega$ then f is an F-Pettis integrable function with $\overline{X}$-valued integrals.*

Proof. Let $\eta \in F$. Since $\eta \circ f$ is continuous, we have that $\eta \circ f$ is F-measurable. Further,

$$\int_\Omega |\eta \circ f(t)| d\mu(t) \leq \|\eta\| \int_\Omega g(t) d\mu(t) < \infty$$

so f is F-Dunford integrable. Therefore, by Proposition 1.2.3, for all $E \in \Sigma$ there exists $B_E \in F^*$ such that

$$B_E(\eta) = \int_E \eta \circ f(t) d\mu(t), \quad \text{for all } \eta \in F.$$

In order to show f is in fact an $\overline{X}$-valued F-Pettis integrable function it suffices to prove B_E is $\sigma(F, \overline{X})$-continuous. For the proof of this fact we separate two cases.

In the first case we assume that the measure μ has compact support R. We need to define the Mackey topology on F, which we'll denote by $\tau(F, X)$. The Mackey topology $\tau(F, X)$ is the weakest topology on F defined by the seminorms

$$\eta \mapsto \sup_{c \in K} |\eta(c)|$$

for all convex, $\sigma(X, F)$-compact, circled subsets K of X. The Mackey topology $\tau(F, X)$ is stronger than the weak topology $\sigma(F, X)$, and by the Mackey-Arens theorem, (see [37, pg. 131]), $\tau(F, X)$ is the strongest topology on F such that the set of continuous functionals on $(F, \tau(F, X))$ is equal to X.

Since $\eta \circ f : \Omega \to \mathbb{C}$ is continuous for every $\eta \in F$ we obtain that if $(U_i)_{i \in I}$ is a cover of $f(R)$ by $\sigma(X, F)$ basic open sets then $(f^{-1}(U_i))_{i \in I}$ is a cover of R by open sets. Since R is compact we obtain a finite subcover $(f^{-1}(U_i))_{i \in J}$ (for some finite subset J of I) of R. Thus $(U_i)_{i \in J}$ is a finite cover of $f(R)$. Hence $f(R)$ is $\sigma(X, F)$-compact.

It is easy to enlarge the set $f(R)$ and produce a circled $\sigma(X, F)$ compact subset of X. Indeed let S be the unit circle in $\mathbb{C}$ and let $\psi : S \times X \to X$ defined by $\psi(\gamma, x) = \gamma x$. Obviously ψ is continuous when S is equipped with its relative topology of the complex numbers and X is equipped with $\sigma(X, F)$. Thus $\psi(S \times f(R))$ is a circled $\sigma(X, F)$-compact subset of X which contains $f(R)$.

The set $\psi(S \times f(R))$ may not be convex but we can enlarge it to produce a convex, circled, and $\sigma(X, F)$-compact subset of X. Indeed, since (X, F) has the Krein-Smulian property, we have that the convex hull K_0 of $\psi(S \times F(R))$ is $\sigma(X, F)$-compact. Obviously K_0 is also circled and convex.

For $\eta \in F$,

$$|B_E(\eta)| = \left| \int_E \eta \circ f(t) d\mu(t) \right| \leq \int_R |\eta \circ f(t)| d\mu(t) \leq \mu(R) \sup_{t \in R} |\eta \circ f(t)| \leq \mu(R) \sup_{x \in K_0} |\eta(x)|.$$

Hence, B_E is $\tau(F, X)$-continuous and thus, by the Mackey-Arens Theorem, B_E is $\sigma(F, X)$-continuous. Therefore $B_E \in X \subseteq \overline{X}$.

In the second case suppose that μ does not have compact support. Since μ is σ-compact, there exists an increasing sequence of compact sets R_n such that $R_n \nearrow \Omega$. Then, for each $n \in \mathbb{N}$, there exists $B_n \in X$ such that

$$\eta(B_n) = \int_{E \cap R_n} \eta \circ f(t) d\mu(t), \quad \text{for all } \eta \in F.$$

For $\eta \in F$ we have that

$$\begin{aligned} |\eta(B_n) - B_E(\eta)| &= \left| \int_{E \cap R_n} \eta \circ f(t) d\mu(t) - \int_E \eta \circ f(t) d\mu(t) \right| \leq \int_{E \setminus R_n} |\eta \circ f(t)| \mu(t) \\ &\leq \|\eta\| \int_{\Omega \setminus R_n} \|f(t)\| d\mu(t) \leq \|\eta\| \int_{\Omega \setminus R_n} g(t) d\mu(t). \end{aligned} \tag{1.8}$$

Thus, for $n, m \in \mathbb{N}$ with $m < n$,

$$|\eta(B_n) - \eta(B_m)| \leq 2\|\eta\| \int_{\Omega \setminus R_m} g(t) d\mu(t).$$

Hence, since F norms X,

$$\|B_n - B_m\| = \sup_{\substack{\eta \in F \\ \|\eta\| \leq 1}} |\eta(B_n - B_m)| \leq 2 \int_{\Omega \setminus R_m} g(t) d\mu(t) \to 0$$

as $m \to \infty$ since $g \in L_1(\Omega)$. Thus, $(B_n)_{n \in \mathbb{N}}$ is Cauchy in X and therefore there exists $B \in \overline{X}$ such that $B_n \to B$, hence $\eta(B_n) \to \eta(B)$ for all $\eta \in F$ (obviously η extends continuously to $B \in \overline{X}$). Note also that by (1.8) and the fact that $g \in L_1(\Omega)$,

$$\eta(B_n) \to B_E(\eta), \quad \text{for all } \eta \in F.$$

Thus, $B_E(\eta) = \eta(B)$ for all $\eta \in F$ and therefore $B_E \in \overline{X}$. □

In [21] equivalent conditions are given for the Krein-Smulian property for real Banach spaces. There are many papers in the literature discussing this property, for example, in [5], it is proven that if X is a real Banach space which does not contain a copy of $\ell_1[0,1]$ and F is a norming subset of the unit ball of X^*, then (X, F) has the Krein-Smulian property.

Next, we look at another sufficient condition for a Banach space valued function to be F-Pettis integrable. We begin with the definition of a Mazur space which was introduced in [42].

Definition 1.2.8. *A **Mazur space** is a locally convex topological vector space (X, τ), where τ is the topology on the set X, such that every sequentially continuous linear functional $f : (X, \tau) \to \mathbb{C}$ is continuous.*

There are several papers in the literature that study implications of the fact that the dual X^* of a Banach space X is a Mazur space with respect to the weak* topology to the Pettis integrability of X-valued functions. See for example [17], [24], and [38]. In the next result rather than considering the weak* topology on X^* we consider the $\sigma(F, X)$ topology on a linear subspace F of X^*.

Theorem 1.2.9. *Let X be a Banach space and let F be a linear subspace of X^* which separates points in X such that $(F, \sigma(F, X))$ is a Mazur space. Let (Ω, Σ, μ) be a measure space. If $f : \Omega \to X$ is F-measurable and $\|f(t)\| \leq g(t)$ for μ-almost all $t \in \Omega$ for some $g \in L_1(\Omega)$ then f is an F-Pettis integrable function.*

Proof. Let $E \in \Sigma$. Since f is F-measurable and $\|f(t)\| \leq g(t)$ for μ-almost all $t \in \Omega$ and $g \in L_1(\Omega)$, by Proposition 1.2.3 there exists $B_E \in F^*$ such that

$$B_E(\eta) = \int_E \eta \circ f(t) d\mu(t) \quad \text{for all } \eta \in F.$$

Next we verify that B_E is $\sigma(F, X)$ sequentially continuous on F. Let $(\eta_n)_{n \in \mathbb{N}}$ be a sequence in F such that $\eta_n \to \eta$ in the $\sigma(F, X)$ topology, for some $\eta \in F$. By the Uniform Boundedness Principle we have that $M = \sup_{n\in\mathbb{N}} \|\eta_n\| < \infty$. Further, $\eta_n(x) \to \eta(x)$ for all $x \in X$, hence, $\eta_n \circ f(t) \to \eta \circ f(t)$ for all $t \in \Omega$. Then we have that $|\eta_n \circ f(t)| \leq Mg(t)$ for μ-almost all $t \in \Omega$ and $g \in L_1(\Omega)$ so, by the Dominated Convergence Theorem, we have that

$$\int_E \eta_n \circ f(t) d\mu(t) \to \int_E \eta \circ f(t) d\mu(t)$$

i.e. $B_E(\eta_n) \to B_E(\eta)$. Hence, B_E is $\sigma(F, X)$ sequentially continuous on F and since $(F, \sigma(F, X))$ is a Mazur space we have, by [27, Theorem 1.3.1] that $B_E \in X$. Therefore, f is an F-Pettis integrable function. □

Next, we give a result which yields an immediate application for Theorem 1.2.9.

Proposition 1.2.10. *If X is a separable Banach space and F is a linear subspace of X^* then $(F, \sigma(F, X))$ is a Mazur space.*

Proof. Suppose ϕ is a $\sigma(F, X)$ sequentially continuous linear functional on F. We want to show that ϕ is $\sigma(F, X)$ continuous. Since X is a Banach space, by [27, Exercise 1.9.14], it is enough to show ϕ restricted to the unit ball $Ba(F)$ of F is $\sigma(F, X)$ continuous. By [15, page 426], $Ba(F)$ with the $\sigma(F, X)$ topology is metrizable so sequential $\sigma(F, X)$ continuity and $\sigma(F, X)$ continuity are equivalent on $Ba(F)$. Thus ϕ restricted to $Ba(F)$ is $\sigma(F, X)$ continuous. □

Corollary 1.2.11. *Let X be a separable Banach space and let F be a norm-closed subspace of X^*. If (Ω, Σ, μ) is a measure space, $f : \Omega \to X$ is F-measurable, and $\|f(t)\| \leq g(t)$ for μ-almost all $t \in \Omega$ for some $g \in L_1(\Omega)$ then f is an F-Pettis integrable function.*

Proof. Follows immediately from Theorem 1.2.9 and Proposition 1.2.10.

□

The next result will be used in Chapter 3 (in the proof of Lemma 3.1.8).

Proposition 1.2.12. *Let X be a normed space, F be a linear subspace of X^* which separates points in X (thus $\sigma(X,F)$ is a Hausdorff topology), f be an X-valued F-Pettis integrable function, and let $T: X \to X$ be a $\sigma(X,F)-\sigma(X,F)$ continuous linear operator. Then $T \circ f$ is an F-Pettis integrable function with X-valued integrals and*

$$T\left((P)-\int_E f(t)d\mu(t)\right) = (P)-\int_E T(f(t))d\mu(t), \quad \textit{for all } E \in \Sigma.$$

Proof. Let $E \in \Sigma$. Since f is an X-valued F-Pettis integrable function, there exists $B_E \in X$ such that

$$\eta(B_E) = \int_E \eta(f(t))d\mu(t), \quad \text{for all } \eta \in F.$$

Since F is a linear subspace of X^* and T is $\sigma(X,F)-\sigma(X,F)$ continuous we have that $\eta \circ T \in F$ for all $\eta \in F$. Thus

$$(\eta \circ T)(B_E) = \int_E (\eta \circ T)(f(t))d\mu(t)$$

i.e.,

$$\eta(T(B_E)) = \int_E \eta(T(f(t)))d\mu(t), \quad \text{for all } \eta \in F.$$

Hence $T \circ f$ is F-Pettis integrable and

$$T(B_E) = (P)-\int_E T \circ f(t)d\mu(t)$$

i.e.,

$$T\left((P)-\int_E f(t)d\mu(t)\right) = (P)-\int_E T(f(t))d\mu(t).$$

□

Weaker and well-established versions of the results of this chapter will be used in the next chapter but we will see applications of the stronger versions in the chapter thereafter.

Next, we want to give some background about quantum Markov semigroups.

1.3 Introduction to quantum Markov semigroups

In this section we provide the necessary definitions and mathematical background that is needed about quantum Markov semigroups. Throughout this section, $\mathcal{H}$ will denote a Hilbert space. To avoid confusion we want to mention from the start that all of our inner products are linear in the second coordinate and conjugate linear in the first. Also, for $x, y \in \mathcal{H}$, we define the rank one operator $|x\rangle\langle y| : \mathcal{H} \to \mathcal{H}$ by $|x\rangle\langle y|(h) = \langle y, h\rangle x$. We will extensively use the σ -weak topology so it is worth recalling: On a general von Neumann algebra, the σ-weak topology is the w^* topology given by its predual (every von Neumann algebra has a predual). If the von Neumann algebra under consideration is $\mathcal{B}(\mathcal{H})$ then the predual is given by the space of all trace class operators on $\mathcal{H}$ which we'll denote by $L_1(\mathcal{H})$. For a detailed description of the duality between $\mathcal{B}(\mathcal{H})$ and $L_1(\mathcal{H})$ we refer the reader to the book of Pedersen [33, Theorem 3.4.13].

Definition 1.3.1. *Let $\mathfrak{A}$ be a von Neumann algebra and let M_n be the set of all $n \times n$ matrices with complex coefficients. Then the algebraic tensor product $\mathfrak{A} \otimes M_n$ can be represented as the *-algebra of $n \times n$ matrices with entries in $\mathfrak{A}$. Every element $A \in \mathfrak{A} \otimes M_n$ can be written in the form*

$$A = \sum_{i,j=1}^{n} A_{ij} \otimes E_{ij}$$

where E_{ij} is the $n \times n$ matrix with 1 in the (i,j)th position and zero everywhere else. If $\mathfrak{B}$ is also a von Neumann algebra and $T : \mathfrak{A} \to \mathfrak{B}$ is a linear operator then we define the linear map $T^{(n)} : \mathfrak{A} \otimes M_n \to \mathfrak{B} \otimes M_n$ by

$$T^{(n)} \left(\sum_{i,j=1}^{n} A_{ij} \otimes E_{ij} \right) = \sum_{i,j=1}^{n} T(A_{ij}) \otimes E_{ij}.$$

*We say a map $T : \mathfrak{A} \to \mathfrak{B}$ is **positive** if it maps positive elements to positive elements. It is called **completely positive** if $T^{(n)}$ is positive for all $n \in \mathbb{N}$. In the case that $\mathfrak{B}$*

acts on a Hilbert space $\mathcal{H}$ it can be proven that T is completely positive if

$$\sum_{i,j=1}^{n} \langle h_i, T(A_i^* A_j) h_j \rangle \geq 0$$

for all $n \in \mathbb{N}$, $A_1, \ldots, A_n \in \mathfrak{A}$, and $h_1, \ldots, h_n \in \mathcal{H}$ [18, Proposition 2.9].

Definition 1.3.2. *A family $(T_t)_{t\geq 0}$ of bounded linear operators on a normed space X is called a* ***semigroup*** *if it satisfies the following properties:*
(i) $T_0 = 1$, and
(ii) $T_{t+s} = T_t T_s$ for all $s, t \geq 0$.

Definition 1.3.3. *Let $\mathfrak{A}$ be a von Neumann algebra. A* ***quantum dynamical semigroup (QDS)*** *is a semigroup $(T_t)_{t\geq 0}$ of σ-weakly continuous, completely positive, linear operators on $\mathfrak{A}$ such that for a fixed $A \in \mathfrak{A}$, the map $t \mapsto T_t(A)$ is σ-weakly continuous.*
Further, if $T_t(1) = 1$ for all $t \geq 0$ then we say the quantum dynamical semigroup is ***Markovian*** *or we simply refer to it as a* ***quantum Markov semigroup (QMS)****.*
If the map $t \mapsto T_t$ is norm continuous then we say the semigroup is ***uniformly continuous****.*

Note 1: If $(T_t)_{t\geq 0}$ is a quantum Markov semigroup then $\|T_t\| = 1$ for all $t \geq 0$. This is due to Dye and Russo [16, Corollary 1].
Note 2: Sufficient conditions for a QDS to be Markovian are given in Chebotarev[7] and simplified sufficient conditions are given in Chebotarev [8] and Chebotarev and Fagnola [9].

Definition 1.3.4. *Given a QDS $(T_t)_{t\geq 0}$, we say that an element $A \in \mathfrak{A}$ belongs to the domain of the infinitesimal generator L of $(T_t))_{t\geq 0}$, denoted by $D(L)$, if*

$$\lim_{t\to 0} \frac{1}{t}(T_t A - A)$$

converges in the σ-weak topology and, in this case, define the ***infinitesimal generator*** *to be the generally unbounded operator L such that*

$$L(A) = \sigma\text{-}weak\text{-}\lim_{t\to 0} \frac{1}{t}(T_t A - A) \quad , \quad A \in D(L).$$

If $(T_t)_{t\geq 0}$ is uniformly continuous then the generator L is bounded and given by

$$L = \lim_{t\to 0} \frac{1}{t}(T_t - 1)$$

where the limit is taken in the norm topology.

It has been proven (see, for example Bratteli and Robinson [4, Proposition 3.1.6]) that the domain of the generator L of a QDS is σ-weakly dense. However, if the QDS is not uniformly continuous then the generator L does not have full domain. Indeed, it is known (see Bratteli and Robinson [4, Proposition 3.1.6]) that L is σ-weakly closed so if L has full domain then it would be bounded. In this case the QDS is then uniformly continuous (see Hille and Phillips [22]).

We have casually mentioned the results of Stinespring [39] and Kraus [29] earlier. Since we will attempt to generalize both, we feel it is necessary to give complete statements of them.

Theorem 1.3.5 (Stinespring)**.** *Let $\mathfrak{B}$ be a C^*-subalgebra of the algebra of all bounded operators on a Hilbert space $\mathcal{H}$ and let $\mathfrak{A}$ be a C^*-algebra with unit. A linear map $T : \mathfrak{A} \to \mathfrak{B}$ is completely positive if and only if it has the form*

$$T(A) = V^* \pi(A) V \tag{1.9}$$

where $(\pi, \mathcal{K})$ is a unital *-representation of $\mathfrak{A}$ on some Hilbert space $\mathcal{K}$, and V is a bounded operator from $\mathcal{H}$ to $\mathcal{K}$.

Theorem 1.3.6 (Kraus)**.** *Let $\mathfrak{A}$ be a von Neumann algebra of operators on a Hilbert space $\mathcal{H}$ and let $\mathcal{K}$ be another Hilbert space. A linear map $T : \mathfrak{A} \to \mathcal{B}(\mathcal{K})$ is normal* *and completely positive if and only if it can be represented in the form*

$$T(A) = \sum_{j=1}^{\infty} V_j^* A V_j \tag{1.10}$$

where $(V_j)_{j=1}^{\infty}$ *is a sequence of bounded operators from* $\mathcal{K}$ *to* $\mathcal{H}$ *such that the series* $\sum_{j=1}^{\infty} V_j^* A V_j$ *converge strongly.*

Chapter 2

Generators of quantum Markov semigroups

In this chapter we look at some existing results for the generators of uniformly continuous QMSs then give analogous results in the case when the QMSs are not uniformly continuous. Lastly, we look at three examples to clarify our findings.

2.1 Generators of uniformly continuous quantum Markov semigroups on $\mathcal{B}(\mathcal{H})$

In this section we recall some results for the form of the generator of a uniformly continuous QMS (which motivate our work on the consequent sections) and we improve existing results. As a motivation for Lindblad's result we start by describing a simple example of a QDS and its generator which comes from Fagnola [18, Example 3.1].

Example 2.1.1. *Let $(U_t)_{t\geq 0}$ be a strongly continuous semigroup on a Hilbert space $\mathcal{H}$. Then, define $T_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, for all $t \geq 0$, by*

$$T_t(A) = U_t A U_t^*$$

Then $(T_t)_{t\geq 0}$ is a quantum dynamical semigroup. Further, if G is the generator of $(U_t)_{t\geq 0}$ and G is bounded then the generator, L, of $(T_t)_{t\geq 0}$ is given by

$$L(A) = GA + AG^*.$$

This form should be compared with (2.1) of Theorem 2.1.3 (Lindblad's result). In Theorem 2.1.3 we give a proof of Lindblad for the case of QMSs defined on $\mathcal{B}(\mathcal{H})$ which allows for a great deal of possibilities for the operator G in the formula of L

which appears in Section 1.1. The following result has been proven in Fagnola [18, Lemma 3.13] for the case of uniformly continuous QDS. Here we remove the uniform continuity assumption.

Proposition 2.1.2. *If L is the generator of a QDS on $\mathcal{B}(\mathcal{H})$ and $\mathfrak{A}$ is a *-subalgebra of $\mathcal{B}(\mathcal{H})$ such that $\mathfrak{A} \subseteq D(L)$ then, for all $A_1, \ldots, A_n \in \mathfrak{A}$ and $u_1, \ldots, u_n \in \mathcal{H}$ such that $\sum_{k=1}^{n} A_k u_k = 0$, we have that*

$$\sum_{i,j=1}^{n} \langle u_i, L(A_i^* A_j) u_j \rangle \geq 0.$$

Proof. We start with a **claim**: If $(T_t)_{t\geq 0}$ is a σ-weakly continuous semigroup of positive operators and L is the generator then, for any $A \in \mathfrak{A}$ and $u \in \mathcal{H}$ such that $Au = 0$ we have that $\langle u, L(A^*A)u \rangle \geq 0$.

Indeed, for $u \in \mathcal{H}$ define $T : \mathcal{H} \to \mathcal{H}$ by $Th = \langle u, h \rangle u = |u\rangle\langle u|(h)$. Clearly T is rank one and hence T is a trace class operator on $\mathcal{H}$. Further, if φ_T is the image of T in $\mathfrak{A}^*$ under the trace duality then

$$\varphi_T(B) = tr(BT) = \langle u, Bu \rangle$$

for all $B \in \mathfrak{A}$. Then, for $A \in \mathfrak{A}$ such that $Au = 0$ we have

$$\langle u, L(A^*A)u \rangle = \varphi_T(L(A^*A)) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \varphi_T(T_\epsilon(A^*A) - A^*A).$$

Further,

$$\begin{aligned}
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \varphi_T(T_\epsilon(A^*A) - A^*A) &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(\langle u, T_\epsilon(A^*A)u \rangle - \langle u, A^*Au \rangle \right) \\
&= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \langle u, T_\epsilon(A^*A)u \rangle \qquad \text{since } Au = 0 \\
&\geq 0 \qquad \text{since } T_\epsilon \geq 0
\end{aligned}$$

which completes the proof of the claim.

Now, suppose $A_1, \ldots, A_n \in \mathfrak{A}$ and $u_1, \ldots, u_n \in \mathcal{H}$ such that $\sum_{k=1}^{n} A_k u_k = 0$. Since

T_t is completely positive, $T_t^{(n)}$ is positive. So, $(T_t^{(n)})_{t\geq 0}$ is a σ-weakly continuous semigroup of positive operators with generator $L^{(n)}$. Let $A_0 = \sum_{k=1}^{n} A_k \otimes E_{1,k}$ and let $u_0 = (u_1, \ldots u_n)^T$ (where T stands for transpose). Then, by the above claim,

$$0 \leq \langle u_0, L^{(n)}(A_0^* A_0) u_0 \rangle = \sum_{j,k=1}^{n} \langle u_j, L(A_j^* A_k) u_k \rangle$$

which completes the proof. □

We will now proceed to look at a proof of Lindblad's Theorem for uniformly continuous QMSs on $\mathcal{B}(\mathcal{H})$. Lindblad's original proof was for any hyperfinite factor in $\mathcal{B}(\mathcal{H})$. The strategy of our proof comes from one utilized by Parthasarathy [32, Proposition 30.14], because it gives us more options in defining the operator G in the formula of L which appears in equation (2.1) below. We make use of the greater flexibility of the form of G in Theorem 2.2.6.

Theorem 2.1.3 (Lindblad)**.** *Let L be the generator of a uniformly continuous QMS on $\mathcal{B}(\mathcal{H})$. Let T be any positive finite rank operator on $\mathcal{H}$. Then there exists $h \in \mathcal{H}$ such that if the operator G is defined on $\mathcal{H}$ by*

$$G(x) = L(|x\rangle\langle Th|)h - \frac{1}{2}\langle h, L(T)h\rangle x$$

then there exists a completely positive map $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ such that

$$L(A) = \phi(A) + GA + AG^* \tag{2.1}$$

for all $A \in \mathcal{B}(\mathcal{H})$.

Proof. By the spectral theorem for compact self-adjoint operators we have that for any positive finite rank operator T there exist finitely many orthonormal vectors $(k'_s)_{s=1}^m$, and positive numbers $(t'_s)_{s=1}^m$ such that $T = \sum_{s=1}^m t'_s |k'_s\rangle\langle k'_s|$. If we define $t = \sum_{s=1}^m t'_s$, $t_s = t'_s/t$, and $k_s = \sqrt{t} k'_s$ then we can rewrite T as $T = \sum_{s=1}^m t_s |k_s\rangle\langle k_s|$ where $t_s \geq 0$, $\sum_{s=1}^m t_s = 1$, and $\langle k_{s_1}, k_{s_2}\rangle = 0$ if $s_1 \neq s_2$. Let $h = \sum_{s=1}^m k_s \|k_s\|^{-2} \in \mathcal{H}$.

Then $\langle h, k_s\rangle = 1$ for all $s = 1, \dots, m$.

Claim: For $s = 1, \dots, m$ if we define the operator $G_s : \mathcal{H} \to \mathcal{H}$ by

$$G_s(x) = L(|x\rangle\langle k_s|)h - \frac{1}{2}\langle h, L(|k_s\rangle\langle k_s|)h\rangle x \tag{2.2}$$

and $\phi_s : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by

$$\phi_s(A) = L(A) - G_s A - A G_s^*$$

then ϕ_s is completely positive.

Once the claim is proved, then the map $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ defined by $\phi = \sum_{s=1}^m t_s \phi_s$ is completely positive since the coefficients t_s are non-negative. Since $\sum_{s=1}^m t_s = 1$, we have that

$$\phi(A) = L(A) - \left(\sum_{s=1}^m t_s G_s\right) A - A\left(\sum_{s=1}^m t_s G_s^*\right) \tag{2.3}$$

and

$$\sum_{s=1}^m t_s G_s^* = \left(\sum_{s=1}^m t_s G_s\right)^*.$$

Hence, if we set $G = \sum_{s=1}^m t_s G_s$, (2.3) gives (2.1). Note that by multiplying (2.2) by t_s and summing up we obtain

$$\begin{aligned} G(x) = \left(\sum_{s=1}^m t_s G_s\right)(x) &= L\left(|x\rangle\langle \sum_{s=1}^m t_s k_s|\right) h - \frac{1}{2}\langle h, L\left(\sum_{s=1}^m t_s |k_s\rangle\langle k_s|\right) h\rangle x \\ &= L\left(|x\rangle\langle Th|\right) h - \frac{1}{2}\langle h, L(T)h\rangle x. \end{aligned}$$

Thus it only remains to prove the claim. Fix $s \in \{1, \dots, m\}$. We vary the technique of [18, Theorem 3.14] as follows. Let $A_1, \dots, A_n \in \mathcal{B}(\mathcal{H})$ and $h_1, \dots, h_n \in \mathcal{H}$. Let $v = -\sum_{i=1}^n A_i h_i$, $A_{n+1} = |v\rangle\langle k_s|$ and $h_{n+1} = h$. Then, since $\langle h, k_s\rangle = 1$,

$$\sum_{i=1}^{n+1} A_i h_i = \sum_{i=1}^n A_i h_i + A_{n+1} h_{n+1} = -v + |v\rangle\langle k_s|(h) = -v + v = 0.$$

Since L is the generator of a uniformly continuous QMS, by Proposition 2.1.2,

$$
\begin{aligned}
0 &\leq \sum_{i,j=1}^{n+1} \langle h_i, L(A_i^* A_j) h_j \rangle \\
&= \sum_{i,j=1}^{n} \langle h_i, L(A_i^* A_j) h_j \rangle + \sum_{i=1}^{n} \langle h_i, L(A_i^* A_{n+1}) h_{n+1} \rangle + \sum_{j=1}^{n} \langle h_{n+1}, L(A_{n+1}^* A_j) h_j \rangle \\
&+ \langle h_{n+1}, L(A_{n+1}^* A_{n+1}) h_{n+1} \rangle.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
0 &\leq \sum_{i,j=1}^{n} \langle h_i, L(A_i^* A_j) h_j \rangle + \sum_{i=1}^{n} \langle h_i, L(|A_i^*(v)\rangle\langle k_s|) h \rangle + \sum_{j=1}^{n} \langle h, L(|k\rangle\langle A_j^*(v)|) h_j \rangle \\
&+ \|v\|^2 \langle h, L(|k_s\rangle\langle k_s|) h \rangle \\
&= \sum_{i,j=1}^{n} \langle h_i, L(A_i^* A_j) h_j \rangle - \sum_{i,j=1}^{n} \langle h_i, L(|A_i^* A_j h_j\rangle\langle k_s|) h \rangle - \sum_{i,j=1}^{n} \langle h, L(|k_s\rangle\langle A_j^* A_i h_i|) h_j \rangle \\
&+ \sum_{i,j=1}^{n} \langle A_i h_i, A_j h_j \rangle \langle h, L(|k_s\rangle\langle k_s|) h \rangle.
\end{aligned}
$$

If we break up the last term into two equal pieces and subtract each from the second and third term of the last expression, then we obtain

$$
\begin{aligned}
0 \leq \sum_{i,j=1}^{n} \Big[&\langle h_i, L(A_i^* A_j) h_j \rangle - \Big(\langle h_i, L(|A_i^* A_j h_j\rangle\langle k_s|) h \rangle - \frac{1}{2} \langle h_i, A_i^* A_j h_j \rangle \langle h, L(|k_s\rangle\langle k_s|) h \rangle \Big) \\
&- \Big(\langle L(|A_j^* A_i h_i\rangle\langle k_s|) h, h_j \rangle - \frac{1}{2} \langle A_j^* A_i h_i, h_j \rangle \langle h, L(|k_s\rangle\langle k_s|) h \rangle \Big) \Big].
\end{aligned}
$$

Define an operator $G_s : \mathcal{H} \to \mathcal{H}$ by $G_s(x) = L(|x\rangle\langle k_s|)h - \frac{1}{2}\langle h, L(|k_s\rangle\langle k_s|)(h)\rangle x$ to continue

$$
\begin{aligned}
0 &\leq \sum_{i,j=1}^{n} \Big(\langle h_i, L(A_i^* A_j) h_j \rangle - \langle h_i, G_s A_i^* A_j h_j \rangle - G_s A_j^* A_i h_i, h_j \rangle \Big) \\
&= \sum_{i,j=1}^{n} \langle h_i, \left(L(A_i^* A_j) - G_s A_i^* A_j - A_i^* A_j G_s^* \right) h_j \rangle = \sum_{i,j=1}^{n} \langle h_i, \phi_s(A_i^* A_j) h_j \rangle
\end{aligned}
$$

which finishes the proof of the claim and the theorem.

□

Definition 2.1.4. *Let T be a positive finite rank operator in $\mathcal{B}(\mathcal{H})$. Then T can be written in the form $T = \sum_{s=1}^{m} t_s |k_s\rangle\langle k_s|$ where $t_s \geq 0$ and $\sum_{s=1}^{m} t_s = 1$. A vector h is called an* ***associate*** *vector for T if and only if $\langle h, k_s \rangle = 1$ for all $s = 1, \ldots, m$.*

2.2 Generators of general quantum Markov semigroups on $\mathcal{B}(\mathcal{H})$

In this section we prove analogous expressions of (2.1) and (1.9) for the generator of a general QMS on $\mathcal{B}(\mathcal{H})$. The main result of the section is Corollary 2.2.9. Heading in this direction, we start with the following:

Theorem 2.2.1. *Let L be the generator of a QMS on $\mathcal{B}(\mathcal{H})$. Then there exists a family $(L_\epsilon)_{\epsilon>0}$ of generators of uniformly continuous QMSs on $\mathcal{B}(\mathcal{H})$ such that*

$$L(A) = \lim_{\epsilon\to 0} L_\epsilon(A)$$

for all $A \in D(L)$, where the limit is taken in the σ-weak topology. Thus, by Theorem 2.1.3, there exists a family $(\phi_\epsilon)_{\epsilon>0}$ of normal completely positive operators on $\mathcal{B}(\mathcal{H})$ and a family $(G_\epsilon)_{\epsilon>0}$ of bounded operators on $\mathcal{H}$ such that

$$L_\epsilon(A) = \phi_\epsilon(A) + G_\epsilon A + AG_\epsilon^*$$

for all $A \in \mathcal{B}(\mathcal{H})$.

Proof. Let L be the generator for a quantum Markov semigroup $(U_t)_{t\geq 0}$. Let $L_\epsilon = L(1-\epsilon L)^{-1}$. Then, for $\epsilon > 0$, L_ϵ is bounded and σ-weakly continuous, since by Proposition 3.1.4 and Proposition 3.1.6 of Bratteli and Robinson [4], $(1-\epsilon L)^{-1}$ is bounded and σ-weakly continuous and

$$L(1-\epsilon L)^{-1} = -\frac{1}{\epsilon}\left(1-(1-\epsilon L)^{-1}\right). \tag{2.4}$$

Define $U_{t,\epsilon} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by $U_{t,\epsilon} = \exp(tL_\epsilon)$. Then we know $(U_{t,\epsilon})_{t\geq 0}$ is a uniformly continuous semigroup. Further, we claim that $(U_{t,\epsilon})_{t\geq 0}$ is contractive. Indeed, by Bratteli and Robinson [4, Theorem 3.1.10] we have that $\|(1-\epsilon L)^{-1}\| \leq 1$ for all $\epsilon > 0$, so

$$\begin{aligned} \|U_{t,\epsilon}\| = \|e^{tL_\epsilon}\| &\leq e^{-t/\epsilon}\sum_{n=0}^{\infty}\frac{(t/\epsilon)^n}{n!}\|(1-\epsilon L)^{-n}\| && \text{by (2.4)} \\ &\leq e^{-t/\epsilon}\sum_{n=0}^{\infty}\frac{(t/\epsilon)^n}{n!} = 1 \end{aligned}$$

and so $(U_{t,\epsilon})_{t\geq 0}$ is contractive. Further, since L_ϵ is σ-weakly continuous we have, by Fagnola [18, Proposition 3.9], that $U_{t,\epsilon}$ is σ-weakly continuous. Also, since $(U_t)_{t\geq 0}$ is Markovian, $1 \in D(L)$ and $L(1) = 0$ so

$$L_\epsilon(1) = L(1-\epsilon L)^{-1}(1) = (1-\epsilon L)^{-1}L(1) = 0.$$

Hence

$$U_{t,\epsilon}(1) = 1 + \sum_{n=1}^{\infty} \frac{t^n}{n!} L_\epsilon^n(1) = 1.$$

So, $\|U_{t,\epsilon}\| = 1$ and the norm is attained at 1 so, by Dye and Russo [16, Corollary 1], $U_{t,\epsilon}$ is positive. Now $(U_t^{(n)})_{t\geq 0}$ is also a quantum Markov semigroup with generator $L^{(n)}$ so, following the above with $U_t^{(n)}$ in place of U_t and $L^{(n)}$ in place of L we get that $\exp\left(tL^{(n)}(1-\epsilon L^{(n)})^{-1}\right) \geq 0$ for all $n \in \mathbb{N}$. We now claim that $L^{(n)}(1-\epsilon L^{(n)})^{-1} = (L(1-\epsilon L)^{-1})^{(n)}$ which will prove that $U_{t,\epsilon}$ is completely positive, since $(L(1-\epsilon L)^{-1})^{(n)}$ is the generator of the semigroup $(U_{t,\epsilon}^{(n)})_{t\geq 0}$. Indeed, for $[A_{i,j}]_{i,j=1,\ldots n} \in D(L) \otimes M_n(\mathbb{C})$,

$$(1-\epsilon L^{(n)})([A_{i,j}]_{i,j=1,\ldots n}) = [(1-\epsilon L)(A_{i,j})]_{i,j=1,\ldots n}$$

hence

$$\begin{aligned}((1-\epsilon L)^{-1})^{(n)}(1-\epsilon L^{(n)})([A_{i,j}]_{i,j=1,\ldots n}) &= [(1-\epsilon L)^{-1}(1-\epsilon L)(A_{i,j})]_{i,j=1,\ldots n} \\ &= [A_{i,j}]_{i,j=1,\ldots n},\end{aligned}$$

which proves that $(1-\epsilon L^{(n)})^{-1} = ((1-\epsilon L)^{-1})^{(n)}$. Hence,

$$L^{(n)}(1-\epsilon L^{(n)})^{-1} = L^{(n)}((1-\epsilon L)^{-1})^{(n)} = (L(1-\epsilon L)^{-1})^{(n)}.$$

Therefore $U_{t,\epsilon}$ is completely positive for all $t \geq 0$ and $\epsilon > 0$. Then, by Theorem 2.1.3, there exists a completely positive map ϕ_ϵ and $G_\epsilon \in \mathcal{B}(\mathcal{H})$ such that

$$L_\epsilon(A) = \phi_\epsilon(A) + G_\epsilon A + AG_\epsilon^*$$

for all $A \in \mathcal{B}(\mathcal{H})$. Next, we claim that $L_\epsilon(A) \underset{\epsilon\to 0}{\longrightarrow} L(A)$ in the σ-weak topology for all $A \in D(L)$. Let $A \in \mathcal{B}(\mathcal{H})$. First, we want to show $(1-\epsilon L)^{-1}(A) \underset{\epsilon\to 0}{\longrightarrow} A$ σ-weakly so let η be an element of the predual $L_1(\mathcal{H})$ of $\mathcal{B}(\mathcal{H})$ and $\gamma > 0$. Since $U_t(A) \underset{t\to 0}{\longrightarrow} A$ σ-weakly, choose $\delta > 0$ so that for any $t < \delta$ we have $|\eta(U_t(A) - A)| < \gamma/2$. Hence

$$\int_0^\delta \epsilon^{-1} e^{-t/\epsilon} \left|\eta(U_t(A) - A)\right| dt < \frac{\gamma}{2}.$$

Then,

$$\begin{aligned}
\left|\eta\left((1-\epsilon L)^{-1}(A)\right) - \eta(A)\right| &= \left|\eta\left(\epsilon^{-1}(\epsilon^{-1} - L)^{-1}(A)\right) - \eta(A)\right| \\
&= \left|\int_0^\infty \epsilon^{-1} e^{-t/\epsilon} \eta(U_t(A)) dt - \eta(A)\right| \qquad \text{by [4, Prop.3.1.6]} \\
&\leq \int_\delta^\infty \epsilon^{-1} e^{-t/\epsilon} \left|\eta(U_t(A) - A)\right| dt \\
&+ \int_0^\delta \epsilon^{-1} e^{-t/\epsilon} \left|\eta(U_t(A) - A)\right| dt \\
&\leq 2\|\eta\|\|A\| \int_\delta^\infty \epsilon^{-1} e^{-t/\epsilon} dt + \frac{\gamma}{2} \\
&= 2\|\eta\|\|A\| e^{-\delta/\epsilon} + \frac{\gamma}{2}.
\end{aligned}$$

So pick $\epsilon_0 > 0$ so that for all $0 < \epsilon < \epsilon_0$ we have $e^{-\delta/\epsilon} < \gamma(4\|\eta\|\|A\|)^{-1}$. Then we have that $|\eta((1-\epsilon L)^{-1}(A)) - \eta(A)| < \gamma$ and therefore $(1-\epsilon L)^{-1}(A) \underset{\epsilon\to 0}{\longrightarrow} A$ σ-weakly for all $A \in \mathcal{B}(\mathcal{H})$. So, for $A \in D(L)$, replace A with LA and we then have $L(1-\epsilon L)^{-1}A \underset{\epsilon\to 0}{\longrightarrow} LA$ σ-weakly since $L(1-\epsilon L)^{-1}A = (1-\epsilon L)^{-1}LA$ for any $A \in D(L)$. Hence $L_\epsilon(A) \underset{\epsilon\to 0}{\longrightarrow} L(A)$ σ-weakly for all $A \in D(L)$. Thus

$$L(A) = \sigma\text{-weak-}\lim_{\epsilon\to 0} \left(\phi_\epsilon(A) + G_\epsilon A + A G_\epsilon^*\right)$$

which completes the proof. □

In Theorem 2.2.1, if $A \in D(L^2)$ we actually get that $L_\epsilon(A) \underset{\epsilon\to 0^+}{\longrightarrow} L(A)$ in norm. Indeed, for $A \in D(L)$,

$$\|(1-\epsilon L)^{-1}A - A\| = \|((1-\epsilon L)^{-1} - (1-\epsilon L)^{-1}(1-\epsilon L))A\| = \epsilon\|(1-\epsilon L)^{-1}LA\| \leq \epsilon\|LA\|$$

since $\|(1-\epsilon L)^{-1}\| \leq 1$ for every $\epsilon > 0$ (see [4, Prop. 3.1.10]). So, for $A \in D(L)$

$$\|(1-\epsilon L)^{-1}A - A\| \leq \epsilon\|LA\| \to 0$$

as $\epsilon \to 0$. Hence, if $A \in D(L^2)$ then

$$L_\epsilon(A) = L(1-\epsilon L)^{-1}A = (1-\epsilon L)^{-1}LA \underset{\epsilon\to 0}{\longrightarrow} LA.$$

For a general QMS on $\mathcal{B}(\mathcal{H})$, we would not expect the completely positive part of the representation of the generator to be bounded. This leads us to the following definition:

Definition 2.2.2. *Let U be a subspace of a Hilbert space $\mathcal{H}$. A linear map ϕ from a linear subspace $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ to the set of sesquilinear forms on $U \times U$ is* ***U-completely positive*** *if for any $k \in \mathbb{N}$, any positive operator $A = (A_{i,j})_{i,j=1,\dots,k} \in \mathcal{A} \otimes M_k(\mathbb{C})$ and for all $u_1, \dots, u_k \in U$ we have that*

$$\sum_{i,j=1}^{k} \phi(A_{i,j})(u_i, u_j) \geq 0.$$

We now proceed to give analogous forms to Lindblad's for the generator of a QMS.

Theorem 2.2.3. *Let L be the generator of a QMS on the von Neumann algebra $\mathcal{B}(\mathcal{H})$. Then there exists a linear (not necessarily closed) subspace W of $\mathcal{H}$, a W-completely positive map ϕ from $D(L)$ into the set of sesquilinear forms on $W \times W$, and a linear operator K from W to $\mathcal{H}$ such that*

$$\langle u, L(A)v\rangle = \phi(A)(u,v) + \langle Ku, Av\rangle + \langle u, AKv\rangle$$

for all $A \in D(L)$ and all $u, v \in W$.

Proof. By Proposition 2.2.1 there exists a family $(\phi_\epsilon)_{\epsilon>0}$ of normal completely positive operators on $\mathcal{B}(\mathcal{H})$ and there exists a family $(G_\epsilon)_{\epsilon>0} \subseteq \mathcal{B}(\mathcal{H})$ such that

$$L(A) = \lim_{\epsilon\to 0}\left(\phi_\epsilon(A) + G_\epsilon A + AG_\epsilon^*\right)$$

for all $A \in D(L)$ where the limit is taken in the σ-weak topology. Define $W \subseteq \mathcal{H}$ by

$$W = \{u \in \mathcal{H} : \lim_{\epsilon\to 0}\langle x, G_\epsilon^* u\rangle \text{ exists for all } x \in \mathcal{H}\}.$$

Then define K on W by $Ku = \text{weak-}\lim_{\epsilon\to 0} G_\epsilon^* u$. Then, for $A \in D(L)$,

$$\begin{aligned}\langle u, L(A)v\rangle &= \lim_{\epsilon\to 0}\langle u, (\phi_\epsilon(A) + G_\epsilon A + AG_\epsilon^*)\, v\rangle \\ &= \lim_{\epsilon\to 0}\langle u, \phi_\epsilon(A)v\rangle + \langle Ku, Av\rangle + \langle u, AKv\rangle\end{aligned}$$

for all $u, v \in W$. Further, since $\lim_{\epsilon\to 0}\langle u, \phi_\epsilon(A)v\rangle$ exists for all $A \in D(L)$ and for all $u, v \in W$, define a linear map ϕ from $D(L)$ to the sesquilinear forms on $W \times W$ by

$$\phi(A)(u, v) = \lim_{\epsilon\to 0}\langle u, \phi_\epsilon(A)v\rangle.$$

Let $A = (A_{i,j})_{i,j=1,\ldots,k} \in D(L) \otimes M_k(\mathbb{C})$ be a positive operator and let $u_1, \ldots, u_k \in W$. Since ϕ_ϵ is completely positive, we have that

$$\sum_{i,j=1}^{k} \langle u_i, \phi_\epsilon(A_{i,j})u_j\rangle \geq 0.$$

Since $\langle u, \phi_\epsilon(A)v\rangle \underset{\epsilon\to 0}{\longrightarrow} \phi(A)(u, v)$ for all $A \in D(L)$ and $u, v \in W$ we have that

$$\sum_{i,j=1}^{n} \phi(A_{i,j})(u_i, u_j) \geq 0$$

which proves that ϕ is W-completely positive. □

Remark 2.2.4. *Assume L is the generator of a QMS on $\mathcal{B}(\mathcal{H})$, $T \in D(L)$ is a positive finite rank operator and h is an associate vector for T. Assume also that $|x\rangle\langle Th| \in D(L)$, for all $x \in \mathcal{H}$. Let the operators $(G_\epsilon)_{\epsilon>0}$ be defined as in the proof of Theorem 2.2.3. Then weak-$\lim_{\epsilon\to 0} G_\epsilon(x)$ and weak-$\lim_{\epsilon\to 0} G_\epsilon^*(x)$ exist for all $x \in \mathcal{H}$. Hence the conclusion of Theorem 2.2.3 is valid with $W = \mathcal{H}$.*

Proof. For $x \in \mathcal{H}$, to see that weak-$\lim_{\epsilon\to 0} G_\epsilon(x)$ exists notice that for all $y \in \mathcal{H}$

$$\langle y, G_\epsilon(x)\rangle = \langle y, L_\epsilon(|x\rangle\langle Th|)h\rangle - \frac{1}{2}\langle h, L_\epsilon(T)h\rangle\langle y, x\rangle.$$

Since $T, |x\rangle\langle Th| \in D(L)$, we have by Theorem 2.2.1 that $L_\epsilon(T) \underset{\epsilon\to 0}{\longrightarrow} L(T)$ and

$$L_\epsilon(|x\rangle\langle Th|) \underset{\epsilon\to 0}{\longrightarrow} L(|x\rangle\langle Th|)$$

σ-weakly. Thus $\langle h, L_\epsilon(T)h\rangle \underset{\epsilon\to 0}{\longrightarrow} \langle h, L(T)\rangle$ and

$$\langle h, L_\epsilon(|x\rangle\langle Th|)h\rangle \underset{\epsilon\to 0}{\longrightarrow} \langle h, L(|x\rangle\langle Th|)\rangle.$$

Hence

$$G_\epsilon(x) \underset{\epsilon\to 0}{\longrightarrow} L(|x\rangle\langle Th|)h - \frac{1}{2}\langle h, L(T)h\rangle x$$

weakly. Next, to see that weak-$\lim_{\epsilon\to 0} G^*_\epsilon(x)$ exists for all $y \in \mathcal{H}$, notice that for all $x \in \mathcal{H}$, $\langle G^*_\epsilon(y), x\rangle = \langle y, G_\epsilon(x)\rangle$. □

Note that Theorem 2.2.3 does not specify the size of the subspace W, while Remark 2.2.4 guarantees that $W = \mathcal{H}$ under some rather strong assumptions. Theorem 2.2.6 gives a form of the generator similar to that of Theorem 2.2.3 with the added advantage that the subspace W is replaced by a subspace U which is easy to describe. The easy form of U enables us to verify that it is dense in $\mathcal{H}$ in Examples 2.4.1 and 2.4.2.

Definition 2.2.5. *If L is the generator of a QMS then the **domain algebra** of L is the largest *-subalgebra of the domain of L, $D(L)$, and is shown in [2] to be given by*

$$\mathcal{A} = \{A \in D(L) : A^*A, AA^* \in D(L)\}.$$

Theorem 2.2.6. *Let L be the generator of a QMS on $\mathcal{B}(\mathcal{H})$. Let $D(L)$ denote its domain and $\mathcal{A}$ denote its domain algebra. Assume there exists a nonzero vector $e \in \mathcal{H}$ such that $|e\rangle\langle e| \in D(L)$. Let U be the linear subspace of $\mathcal{H}$ defined by $U = \{x \in \mathcal{H} : |x\rangle\langle e| \in \mathcal{A}\}$. Then there exists a linear map $G : U \to \mathcal{H}$ and a U-completely positive map ϕ from $\mathcal{A}$ to the set of sesquilinear forms on $U \times U$ such that*

$$\langle u, L(A)v\rangle = \phi(A)(u, v) + \langle u, GAv\rangle + \langle GA^*u, v\rangle$$

for all $A \in \mathcal{A}$ and $u, v \in U$. Moreover, if T is any positive finite rank operator in $D(L)$ which has an associate vector h such that $Th = e$ then the linear map $G : U \to \mathcal{H}$

can be chosen to be

$$G(u) = L(|u\rangle\langle e|)h - \frac{1}{2}\langle h, L(T)h\rangle u \quad , \quad u \in U.$$

One such example of T is the rank-one projection $|e\rangle\langle e|$ whose associate vector h is taken to be equal to e.

Remark 2.2.7. *First, for the sake of clarity we explain the definition of U. Note that by Definition 2.2.5, for $x \in \mathcal{H}$, $|x\rangle\langle e| \in \mathcal{A}$ is equivalent to having the following three conditions hold: $|x\rangle\langle e| \in D(L)$, $(|x\rangle\langle e|)^* \circ |x\rangle\langle e| = \|x\|^2|e\rangle\langle e| \in D(L)$, and $|x\rangle\langle e| \circ (|x\rangle\langle e|)^* = \|e\|^2|x\rangle\langle x| \in D(L)$. Thus if U contains non-zero vectors then $|e\rangle\langle e| \in D(L)$ and that is why this condition appears explicitly in the statement of Theorem 2.2.6.*

Proof of Theorem 2.2.6. By Theorem 2.2.1 there exists a family $(L_\epsilon)_{\epsilon>0}$ of generators of uniformly continuous QMSs on $\mathcal{B}(\mathcal{H})$ such that $L(A) = \sigma\text{-weak-}\lim_{\epsilon\to 0^+} L_\epsilon(A)$ for every $A \in D(L)$. Also there exist families of completely positive operators $(\phi_\epsilon)_{\epsilon>0}$ on $\mathcal{B}(\mathcal{H})$ and bounded operators $(G_\epsilon)_{\epsilon>0}$ on $\mathcal{H}$ such that

$$L_\epsilon(A) = \phi_\epsilon(A) + G_\epsilon A + AG_\epsilon^*$$

for all $A \in D(L)$. Let $v \in U$ and let $A \in \mathcal{A}$. Since $\mathcal{A}$ is an algebra, we obtain $|Av\rangle\langle e| = A \circ |v\rangle\langle e| \in \mathcal{A}$. Then, using the explicit form for G_ϵ from Theorem 2.1.3, we have

$$G_\epsilon Av = L_\epsilon(|Av\rangle\langle e|)h - \frac{1}{2}\langle h, L_\epsilon(T)h\rangle Av. \tag{2.5}$$

Since $|Av\rangle\langle e| \in \mathcal{A} \subseteq D(L)$ we obtain by Theorem 2.2.1 that $L_\epsilon(|Av\rangle\langle e|) \underset{\epsilon\to 0}{\longrightarrow} L(|Av\rangle\langle e|)$ in the σ-weak topology. Thus for any $u \in \mathcal{H}$ we obtain

$$\langle u, L_\epsilon(|Av\rangle\langle e|)h\rangle \underset{\epsilon\to 0}{\longrightarrow} \langle u, L(|Av\rangle\langle e|)h\rangle. \tag{2.6}$$

Also, by Theorem 2.2.1, since $T \in D(L)$ we have that $L_\epsilon(T) \underset{\epsilon\to 0}{\longrightarrow} L(T)$ in the σ-weak topology and hence

$$\langle h, L_\epsilon(T)h\rangle \underset{\epsilon\to 0}{\longrightarrow} \langle h, L(T)h\rangle. \tag{2.7}$$

Thus, by (2.5), (2.6), and (2.7), for any $u \in \mathcal{H}$, $v \in U$ and $A \in \mathcal{A}$ we have

$$\begin{aligned}\langle u, G_\epsilon A v\rangle &= \langle u, L_\epsilon(|Av\rangle\langle e|)h\rangle - \frac{1}{2}\langle h, L_\epsilon(T)h\rangle\langle u, Av\rangle \\ &\underset{\epsilon\to 0}{\longrightarrow} \langle u, L(|Av\rangle\langle e|)h\rangle - \frac{1}{2}\langle h, L(T)h\rangle\langle u, Av\rangle = \langle u, GAv\rangle.\end{aligned}$$

Similarly, for $u \in U$, $v \in \mathcal{H}$, and $A \in \mathcal{A}$, we have

$$\langle u, AG_\epsilon^* v\rangle \underset{\epsilon\to 0}{\longrightarrow} \langle GA^* u, v\rangle.$$

Thus for $u, v \in U$ and $A \in \mathcal{A}$,

$$\begin{aligned}\langle u, L(A)v\rangle &= \lim_{\epsilon\to 0}\langle u, L_\epsilon(A)v\rangle = \lim_{\epsilon\to 0}\langle u, \left(\phi_\epsilon(A) + G_\epsilon A + AG_\epsilon^*\right) v\rangle \\ &= \lim_{\epsilon\to 0}\langle u, \phi_\epsilon(A)v\rangle + \langle u, GAv\rangle + \langle GA^* u, v\rangle.\end{aligned}$$

Thus $\lim_{\epsilon\to 0}\langle u, \phi_\epsilon(A)v\rangle$ exists for all $A \in \mathcal{A}$ and for all $u, v \in U$, and therefore define

$$\phi(A)(u, v) = \lim_{\epsilon\to 0}\langle u, \phi_\epsilon(A)v\rangle.$$

Let $A = (A_{i,j})_{i,j=1,\ldots,k} \in \mathcal{A} \otimes M_k(\mathbb{C})$ be a positive operator and let $u_1, \ldots, u_k \in U$. Since ϕ_ϵ is completely positive we have that

$$\sum_{i,j=1}^{k} \langle u_i, \phi_\epsilon(A_{i,j})u_j\rangle \geq 0.$$

Since $\langle u, \phi_\epsilon(A)v\rangle \to \phi(A)(u, v)$ for all $A \in \mathcal{A}$ and $u, v \in U$ we have that

$$\sum_{i,j=1}^{n} \phi(A_{i,j})(u_i, u_j) \geq 0.$$

Therefore ϕ is U-completely positive. □

While restricting to the domain algebra helps us to understand the subspace U and the operator G, it does come at a cost since the domain of the generator is σ-weakly dense while there are examples of QMSs whose domain algebras are not very large. Indeed, F. Fagnola [19] gives an example of a QMS on $\mathcal{B}(L_2(0,\infty),\mathbb{C})$ where $\mathcal{A}$ is not σ-weakly dense in $\mathcal{B}(L_2(0,\infty),\mathbb{C})$. In Section 6 we will look at several examples where U is dense in $\mathcal{H}$ and also verify the above form for the generator L.

We will proceed by showing that we have analogous results to that of Stinespring's. In the next proposition when we say a map $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, where $\mathcal{A}$ is a (not necessarily closed) unital *-subalgebra of $\mathcal{B}(\mathcal{H})$, is a unital *-representation we mean that it is a unital norm-continuous *-homomorphism.

Theorem 2.2.8. *Suppose $\mathcal{A}$ is a unital (not necessarily closed) *-subalgebra of $\mathcal{B}(\mathcal{H})$, U is a (not necessarily closed) linear subspace of $\mathcal{H}$, and ϕ is a U-completely positive map from $\mathcal{A}$ to the set of sesquilinear forms on $U \times U$. Then there exists a Hilbert space $\mathcal{K}$, a unital *-representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ of norm equal to one, and a linear map $V : U \to \mathcal{K}$ such that*

$$\phi(A)(u, w) = \langle Vu, \pi(A)Vw\rangle_{\mathcal{K}}$$

for all $u, w \in U$.

Proof. Define a sesquilinear form $(\cdot, \cdot) : (\mathcal{A} \otimes U) \times (\mathcal{A} \otimes U) \to \mathbb{C}$ by

$$(x, y) = \sum_{i,j=1}^{n} \phi(A_i^* B_j)(u_i, v_j)$$

where $x = \sum_{i=1}^{n} A_i \otimes u_i$ and $y = \sum_{j=1}^{n} B_j \otimes v_j$ (since we allow zero entries, we can have the same upper limit n in both sums). Since ϕ is U-completely positive, $(x, x) \geq 0$ for all $x \in \mathcal{A} \otimes U$ so $(\cdot, \cdot)$ is a positive definite sesquilinear form. For $x \in \mathcal{A} \otimes U$ let $\|x\|_{(\cdot,\cdot)} = \sqrt{(x, x)}$. Let $N = \{x \in \mathcal{A} \otimes U : (x, x) = 0\}$. Since $(\cdot, \cdot)$ is a positive definite sesquilinear form, by the Cauchy-Schwartz inequality, N is a linear subspace of $\mathcal{A} \otimes U$ and we have that the completion of $(\mathcal{A} \otimes U)/N$, which we'll denote by $\mathcal{K}$, is a Hilbert space where the inner product is given by $\langle x + N, y + N\rangle_{\mathcal{K}} = (x, y)$. Let $\pi_0 : \mathcal{A} \to L(\mathcal{A} \otimes U)$ (where $L(X)$ denotes the linear (not necessarily bounded) operators from X to X) defined by

$$\pi_0(A)\left(\sum_{i=1}^{n} A_i \otimes u_i\right) = \sum_{i=1}^{n} AA_i \otimes u_i.$$

Then, for $A \in \mathcal{A}$, $x = \sum_{i=1}^{n} A_i \otimes u_i \in \mathcal{A} \otimes U$ and $y = \sum_{j=1}^{n} B_j \otimes v_j \in \mathcal{A} \otimes U$ we have

$$(x, \pi_0(A)y) = \left(\sum_{i=1}^{n} A_j \otimes u_j, \sum_{j=1}^{n} AB_j \otimes v_j \right) = \sum_{i,j=1}^{n} \phi((A^* A_i)^* B_j)(u_i, v_i) = (\pi_0(A^*)x, y).$$

Fix $x = \sum_{i=1}^{n} A_i \otimes u_i \in \mathcal{A} \otimes U$ and define $\omega : \mathcal{A} \to \mathbb{C}$ by $\omega(A) = (x, \pi_0(A)x)$ for $A \in \mathcal{A}$. Clearly ω is linear. Then for $A \in \mathcal{A}$,

$$\begin{aligned} \omega(A^*A) &= (x, \pi_0(A^*A)x) \\ &= \sum_{i,j=1}^{n} \phi(A_i^* A^* A A_j)(u_i, u_j) \\ &= \sum_{i,j=1}^{n} \phi((AA_i)^*(AA_j))(u_i, u_j) \\ &\geq 0 \end{aligned}$$

since ϕ is U-completely positive. Now, for $A \in \mathcal{A}$, $A^*A \leq \|A^*A\| 1$ since $1 \in \mathcal{A}$. Then, since ω is positive,

$$\omega(A^*A) \leq \|A^*A\| \omega(1) = \|A^*A\| \|x\|^2_{(\cdot,\cdot)}.$$

So,

$$\|\pi_0(A)x\|^2_{(\cdot,\cdot)} = (\pi_0(A)x, \pi_0(A)x) = (x, \pi_0(A^*A)x) = \omega(A^*A) \leq \|A^*A\| \|x\|^2_{(\cdot,\cdot)}. \quad (2.8)$$

Thus in fact $\pi_0(A) \in \mathcal{B}(\mathcal{A} \otimes U)$ (bounded operators from $\mathcal{A} \otimes U$ to $\mathcal{A} \otimes U$) and $\|\pi_0(A)\| \leq \sqrt{\|A^*A\|} = \|A\|$. Hence, if $(x, x) = 0$ then $(\pi_0(A)x, \pi_0(A)x) = 0$ for all $A \in \mathcal{A}$. Now, define $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ by $\pi(A)(x + N) = \pi_0(A)x + N$ which is well-defined since we saw above that $(x, x) = 0 \Rightarrow (\pi_0(A)x, \pi_0(A)x) = 0$. It is obvious that π is linear, $\pi(1_{\mathcal{H}}) = 1_{\mathcal{K}}$, and for $A, B \in \mathcal{A}$ we have $\pi(A^*) = \pi(A)^*$ and $\pi(AB) = \pi(A)\pi(B)$ as in the proof of Stinespring [39, Theorem 1]. Further, let $V : U \to \mathcal{K}$ where $Vu = 1 \otimes u + N$ for all $u \in U$. Then, for $u, w \in U$ and $A \in \mathcal{A}$ we have that

$$\langle Vu, \pi(A)Vw \rangle_{\mathcal{K}} = \langle 1 \otimes u + N, A \otimes w + N \rangle_{\mathcal{K}} = (1 \otimes u, A \otimes w) = \phi(A)(u, w).$$

Any representation of a unital C^*-algebra into another is known to be bounded and in fact have norm equal to one (obtained at the identity, see Sunder [40, Lemma 3.4.2(b)]). The domain algebra $\mathcal{A}$ is not closed so it is not a C^*-algebra but we verify here that the representation π has norm equal to one. Indeed, let $A \in \mathcal{A}$ and $x + N \in \mathcal{K}$. Then

$$\|\pi(A)(x+N)\|_{\mathcal{K}} = \|\pi_0(A)x + N\|_{\mathcal{K}} = \langle \pi_0(A)x + N, \pi_0(A)x + N\rangle^{1/2} = \|\pi_0(A)x\|_{(\cdot,\cdot)}$$

Further, by (2.8),

$$\|\pi_0(A)x\|_{(\cdot,\cdot)} \leq \|A^*A\|^{1/2}\|x\|_{(\cdot,\cdot)} = \|A\|\|x+N\|_{\mathcal{K}}.$$

and therefore $\|\pi(A)\| \leq \|A\|$ for all $A \in \mathcal{A}$ and the proof is complete. □

Theorems 2.2.6 and 2.2.8 bring us one step closer to the explicit form of the generator of a QMS. Our progress is summed up in the following which is the main result of this section.

Corollary 2.2.9. *Let L be the generator of a QMS on the von Neumann algebra $\mathcal{B}(\mathcal{H})$ and let $\mathcal{A}$ be its domain algebra. Assume there exists a nonzero vector $e \in \mathcal{H}$ such that $|e\rangle\langle e| \in D(L)$. Let U be the linear subspace of $\mathcal{H}$ defined by $U = \{x \in \mathcal{H} : |x\rangle\langle e| \in \mathcal{A}\}$. Then there exists a Hilbert space $\mathcal{K}$, a unital *-representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$, and linear maps $G : U \to \mathcal{H}$ and $V : U \to \mathcal{K}$ such that*

$$\langle u, L(A)w\rangle = \langle Vu, \pi(A)Vw\rangle_{\mathcal{K}} + \langle u, GAw\rangle + \langle GA^*u, w\rangle$$

for all $u, w \in U$ and $A \in \mathcal{A}$. Moreover, if T is any positive finite rank operator in $D(L)$ which has an associate vector h such that $Th = e$ then the linear map $G : U \to \mathcal{H}$ can be chosen to be

$$G(u) = L(|u\rangle\langle e|)h - \frac{1}{2}\langle h, L(T)h\rangle u \quad , \quad u \in U.$$

One such example of T is the rank-one projection $|e\rangle\langle e|$ whose associate vector h is taken to be equal to e.

Proof. Follows immediately from 2.2.6 and 2.2.8. □

We do not know if the map G that appears in Theorem 2.2.6 and Corollary 2.2.9 is closed. In Proposition 2.2.10 we define a linear operator $\widehat{G} : U \to \mathcal{B}(\mathcal{H})$ such that $\widehat{G}(x)(h) = G(x)$, for a positive finite rank operator $T \in D(L)$ and h an associate vector of T, and we study its closability.

Proposition 2.2.10. *Let L be the generator of a QMS on the von Neumann algebra $\mathcal{B}(\mathcal{H})$. Suppose there exists a positive, finite rank operator $T \in D(L)$ and an associate vector h for T such that $|Th\rangle\langle Th| \in D(L)$. Let $U = \{x \in \mathcal{H} : |x\rangle\langle Th| \in \mathcal{A}\}$ and define $\widehat{G} : U \to \mathcal{B}(\mathcal{H})$ by*

$$\widehat{G}(x)(v) = L(|x\rangle\langle Th|)v - \frac{1}{2}\langle h, L(T)v\rangle x.$$

Then $\widehat{G}$ is ($\|\cdot\|$, σ-weakly)-closable. Further, if we define $U_0 = \{x \in \mathcal{H} : |x\rangle\langle Th| \in D(L)\}$ then $U \subseteq U_0$ and $\widehat{G}$ defined on U_0 is ($\|\cdot\|$, σ-weakly)-closed.

Proof. Let $(x_n)_{n\geq 1} \subseteq U$ such that $x_n \to 0$ in norm and $\widehat{G}(x_n) \to A \in \mathcal{B}(\mathcal{H})$ σ-weakly. Then, $|x_n\rangle\langle Th| \in \mathcal{A} \subseteq D(L)$. We claim that $|x_n\rangle\langle Th| \underset{n\to 0}{\longrightarrow} 0$ σ-weakly. Indeed, let $(u_k)_{k\geq 1}, (v_k)_{k\geq 1} \subseteq \mathcal{H}$ such that $\sum_k \|u_k\|^2 < \infty$ and $\sum_k \|v_k\|^2 < \infty$. Then

$$\begin{aligned}\left|\sum_{k=1}^{\infty}\langle u_k, |x_n\rangle\langle Th|v_k\rangle\right| &\leq \sum_{k=1}^{\infty} |\langle Th, v_k\rangle\langle u_k, x_n\rangle| \\ &\leq \|x_n\| \left(\sum_{k=1}^{\infty} \|Th\|^2\|v_k\|^2\right)^{1/2} \left(\sum_{k=1}^{\infty} \|u_k\|^2\right)^{1/2} \\ &= c_1\|x_n\|\end{aligned}$$

and since $\|x_n\| \to 0$ we have that $|x_n\rangle\langle Th| \to 0$ σ-weakly. Similarly, we claim that the sequence of bounded linear operators $v \mapsto \langle h, L(T)v\rangle x_n$ (simply denoted as $\langle h, L(T)\cdot\rangle x_n$) converges to 0 σ-weakly as $n \to \infty$. Indeed,

$$\left|\sum_{k=1}^{\infty}\langle u_k, \langle h, L(T)v_k\rangle x_n\rangle\right| \leq c_2\|x_n\| \left(\sum_{k=1}^{\infty} \|u_k\|^2\right)^{1/2} \left(\sum_{k=1}^{\infty} \|v_k\|^2\right)^{1/2}.$$

Since $\langle h, L(T)\cdot\rangle x_n \to 0$ σ-weakly as $n \to \infty$ and $\widehat{G}(x_n) \to A$ σ-weakly we have that $L(|x_n\rangle\langle Th|) \to A$ σ-weakly. Thus, since L is σ-weakly closed on its domain D(L) (see Bratteli and Robinson [4, Theorem 3.1.10]), and $|x_n\rangle\langle Th| \to 0$ σ-weakly we have that $A = L(0) = 0$ and therefore $\widehat{G}$ is closable. For the last statement of Proposition 2.2.10 suppose that $(x_n)_{n\geq 1} \subseteq U_0$ with $x_n \to x$ in norm and $\widehat{G}(x_n) \to A \in \mathcal{B}(\mathcal{H})$ σ-weakly. Repeating the above argument with $x_n - x$ in place of x_n we obtain that $|x_n - x\rangle\langle Th| \to 0$ σ-weakly (hence $|x_n\rangle\langle Th| \to |x\rangle\langle Th|$ σ-weakly), and that $\langle h, L(T)\cdot\rangle(x_n - x) \to 0$ σ-weakly as $n \to \infty$, hence $\langle h, L(T)\cdot\rangle x_n \to \langle h, L(T)\cdot\rangle x$ σ-weakly as $n \to \infty$. Since $\widehat{G}(x_n) \to A$ σ-weakly, we obtain that

$$L(|x_n\rangle\langle Th|) \to A + \frac{1}{2}\langle h, L(T)\cdot\rangle x$$

σ-weakly. Thus, since L is σ-weakly closed on its domain $D(L)$, we obtain that $|x\rangle\langle Th| = A + \frac{1}{2}\langle h, L(T)\cdot\rangle x$, i.e., $\widehat{G}(x) = A$, which proves that $\widehat{G}$ defined on U_0 is $(\|\cdot\|, \sigma\text{-weakly})$-closed. $\square$

As mentioned earlier, we will illustrate the form of the generator L and discuss the subspace U in several examples in Section 2.4 but first we would like to make our first attempt at obtaining an analogous result to that of Kraus' (Theorem 1.3.6). Our second attempt will be given in the next chapter.

2.3 An attempt to extend Kraus' result

Theorems 2.2.6 and 2.2.8 describe the form of the generator of a QMS on $\mathcal{B}(\mathcal{H})$. Unfortunately we do not have a result similar to Theorem 1.3.6 for the form of the representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ which appears in the conclusion of Theorem 2.2.8. For the uniformly continuous QMSs on $\mathcal{B}(\mathcal{H})$, π turns out to be a normal representation on $\mathcal{B}(\mathcal{H})$ and the map V which appears in Theorem 1.3.5 turns out to be bounded.

This section is dedicated to proving, under suitable assumptions, continuity properties of the operators V and ϕ which appear in Theorem 2.2.8 in the hopes of obtaining a dilation for ϕ, similar to Theorem 1.3.6. While we do not achieve this, we get rather close and identify what we see is ultimately needed to finish. We also have some continuity results which are of interest in their own right.

Proposition 2.3.1. *Let L be the generator of a QMS on the von Neumann algebra $\mathcal{B}(\mathcal{H})$. Further, suppose there exists a positive, finite rank operator $T \in D(L)$ and an associate vector h for T such that $|Th\rangle\langle Th| \in D(L)$. Let $U = \{x \in \mathcal{H} : |x\rangle\langle Th| \in \mathcal{A}\}$ and define $G : U \to \mathcal{H}$ by*

$$Gx = L(|x\rangle\langle Th|)h - \frac{1}{2}\langle h, L(T)h\rangle x$$

and $V : U \to \mathcal{K}$ by

$$Vx = 1 \otimes x + N$$

where $\mathcal{K}$ is the Hilbert space given in Theorem 2.2.8. Also, suppose that

$$\text{there exists } C > 0 \text{ such that } \|L(|x\rangle\langle Th|)h\| \leq C\|x\| \text{ for all } x \in U. \tag{2.9}$$

Then G is bounded on U. If the map ϕ of Theorem 2.2.8 satisfies the conclusion of Theorem 2.2.6 then the map V is bounded on U as well.

Proof. For $x \in U$,

$$\|Gx\| = \|L(|x\rangle\langle Th|)h - \frac{1}{2}\langle h, L(T)h\rangle x\| \leq C\|x\| + \frac{1}{2}\,|\langle h, L(T)h\rangle|\,\|x\| \leq C'\|x\|$$

and so G is bounded on U. Further, let $x \in U$. Then

$$\|Vx\|^2_{\mathcal{K}} = \|1 \otimes x + N\|^2_{\mathcal{K}} = (1 \otimes x, 1 \otimes x) = \phi(1)(x, x) = |\phi(1)(x, x)|.$$

Hence, by the conclusion of Theorem 2.2.6, since $L(1) = 0$ we get that

$$|\phi(1)(x, x)| = |-\langle x, Gx\rangle - \langle Gx, x\rangle| \leq C\|x\|^2 + C\|x\|^2.$$

Therefore V is also bounded on U. □

The operator G of Proposition 2.3.1 is the same as in Theorem 2.2.6 and Corollary 2.2.9. The operator V of Proposition 2.3.1 is the same as in Theorem 2.2.8 and Corollary 2.2.9. Corollary 2.2.9 and Proposition 2.3.1 are used in the proof of the next result.

Proposition 2.3.2. *Let L be the generator of a QMS on the von Neumann algebra $\mathcal{B}(\mathcal{H})$ and let $\mathcal{A}$ denote its domain algebra. Further, suppose there exists a positive, finite rank operator $T \in D(L)$ and an associate vector h for T such that $|Th\rangle\langle Th| \in D(L)$. Let $U = \{x \in \mathcal{H} : |x\rangle\langle Th| \in \mathcal{A}\}$. Assume that* (2.9) *is valid and that $\overline{U}^{\|\cdot\|} = \mathcal{H}$. Then, there exist a linear map $G : U \to \mathcal{H}$, a Hilbert space $\mathcal{K}$, a linear map $V : U \to \mathcal{K}$ and a unital *-representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ such that*

$$L(A) = V^*\pi(A)V + GA + AG^* \tag{2.10}$$

for all $A \in \mathcal{A}$. Further, define $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ by $\psi(A) = GA + AG^*$. Then ψ is σ-weakly - σ-weakly continuous. Lastly, the map $\varphi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ defined by $\varphi(A) = V^*\pi(A)V$ is σ-weakly - σ-weakly closable.

Remark 2.3.3. *Note that the assumptions of Proposition 2.3.2 are rather strong since* (2.10) *implies that L is bounded on $\mathcal{A}$ (but not necessarily on $\mathcal{B}(\mathcal{H})$).*

Proof of Prop. 2.3.2. By Corollary 2.2.9 there exist a linear map $G : U \to \mathcal{H}$, a Hilbert space $\mathcal{K}$, a linear map $V : U \to \mathcal{K}$ and a unital *-representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ such that

$$\langle x, L(A)y\rangle = \langle Vx, \pi(A)Vy\rangle + \langle x, GAy\rangle + \langle GA^*x, y\rangle$$

for all $A \in \mathcal{A}$ and $x, y \in U$. By Proposition 2.3.1 and the assumption that $\overline{U}^{\|\cdot\|} = \mathcal{H}$ we see that

$$\langle x, L(A)y\rangle = \langle x, V^*\pi(A)Vy\rangle + \langle x, GAy\rangle + \langle x, AG^*y\rangle$$

for all $A \in \mathcal{A}$ and $x, y \in \mathcal{H}$. Thus $L(A) = V^*\pi(A)V + GA + AG^*$ for all $A \in \mathcal{A}$. Let $(A_\lambda)_\lambda \subseteq \mathcal{B}(\mathcal{H})$ be a net such that $A_\lambda \underset{\lambda}{\to} A$ σ-weakly for some $A \in \mathcal{B}(\mathcal{H})$. Let

$(x_n)_{n\geq 1}, (y_n)_{n\geq 1} \subseteq \mathcal{H}$ such that $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$ and $\sum_{n=1}^{\infty} \|y_n\|^2 < \infty$. Then

$$\begin{aligned}\sum_{n=1}^{\infty}\langle x_n, \psi(A_\lambda)y_n\rangle &= \sum_{n=1}^{\infty}\left(\langle x_n, GA_\lambda y_n\rangle + \langle x_n, A_\lambda G^* y_n\rangle\right)\\ &= \sum_{n=1}^{\infty}\left(\langle G^* x_n, A_\lambda y_n\rangle + \langle x_n, A_\lambda G^* y_n\rangle\right)\\ &\underset{\lambda}{\to} \sum_{n=1}^{\infty}\left(\langle G^* x_n, A y_n\rangle + \langle x_n, A G^* y_n\rangle\right)\end{aligned}$$

since $\sum_{n=1}^{\infty} \|G^* x_n\|^2 < \infty$ and $\sum_{n=1}^{\infty} \|G^* y_n\|^2 < \infty$. So we have that ψ is σ-weakly - σ-weakly continuous.

Next, let $(A_\lambda)_\lambda \subseteq \mathcal{A}$ be a net such that $A_\lambda \underset{\lambda}{\to} 0$ σ-weakly and $\varphi(A_\lambda) \underset{\lambda}{\to} B$ σ-weakly, for some $B \in \mathcal{B}(\mathcal{H})$, where $\varphi(A) = V^*\pi(A)V$. Let $(x_n)_{n\geq 1}, (y_n)_{n\geq 1} \subseteq \mathcal{H}$ such that $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$ and $\sum_{n=1}^{\infty} \|y_n\|^2 < \infty$. Then,

$$\sum_{n=1}^{\infty}\langle x_n, L(A_\lambda)y_n\rangle = \sum_{n=1}^{\infty}\left(\langle x_n, V^*\pi(A_\lambda)Vy_n\rangle + \langle x_n, \psi(A_\lambda)y_n\rangle\right) \to \sum_{n=1}^{\infty}\langle x_n, By_n\rangle$$

since ψ is σ-weakly - σ-weakly continuous and $\varphi(A_\lambda) \to B$ σ-weakly. Then, since L is σ-weakly-σ-weakly closed on its domain $D(L)$ and therefore σ-weakly-σ-weakly closable on $\mathcal{A}$ we have that $B = L(0) = 0$. So we have that φ is σ-weakly - σ-weakly closable. □

If one assumes (2.9) but does not assume that $\overline{U}^{\|\cdot\|} = \mathcal{H}$ then the proof of Proposition 2.3.2 gives the following.

Remark 2.3.4. *Consider the situation described in Proposition 2.3.2 without assuming that $\overline{U}^{\|\cdot\|} = \mathcal{H}$. Let*

$$F = \left\{\sum_{n=1}^{\infty} |x_n\rangle\langle y_n| : (x_n)_{n\geq 1}, (y_n)_{n\geq 1} \subseteq \overline{U}^{\|\cdot\|} \text{ s.t. } \sum_{n=1}^{\infty}\|x_n\|^2 < \infty\,, \sum_{n=1}^{\infty}\|y_n\|^2 < \infty\right\}$$

then $F \subseteq L_1(\mathcal{H})$ (the space of trace class operators on $\mathcal{H}$). Further, if we define $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ by $\psi(A) = GA + AG^*$ then under assumption (2.9), ψ is $\sigma(\mathcal{A}, F) - \sigma(\mathcal{B}(\mathcal{H}), F)$ continuous.

We did not find an application of the above remark.

Definition 2.3.5. *A pair (π, V) satisfying $T(A) = V^*\pi(A)V$ where π is a representation on $\mathcal{A}$ and $V : U \to \mathcal{K}$, is called a* ***minimal Stinespring representation*** *if the set*

$$\{\pi(A)Vu : A \in \mathcal{A}, \quad u \in U\}$$

is total in $\mathcal{K}$.

If we look back at Theorem 2.2.8 to the definitions of ϕ, V, and $\mathcal{K}$, it is easy to see that our (π, V) is a minimal Stinespring representation (as in the proof of the original result of Stinespring [39, Theorem 1]). This will be used in the following result.

Proposition 2.3.6. *Let L be the generator of a QMS on the von Neumann algebra $\mathcal{B}(\mathcal{H})$ and let $\mathcal{A}$ denote its domain algebra. Suppose there exists a positive, finite rank operator $T \in D(L)$ and an associate vector h for T such that $|Th\rangle\langle Th| \in D(L)$. Let $U = \{x \in \mathcal{H} : |x\rangle\langle Th| \in \mathcal{A}\}$. Also, suppose that (2.9) is valid and that $\overline{U}^{\|\cdot\|} = \mathcal{H}$. Then the unital *-representation $\pi : \mathcal{A} \to \mathcal{K}$ which appears in the statement of Proposition 2.3.2, is σ-weakly - σ-weakly closable.*

Proof. Let $(A_\lambda)_\lambda \subseteq \mathcal{A}$ be a net such that $A_\lambda \underset{\lambda}{\longrightarrow} 0$ σ-weakly and $\pi(A_\lambda) \underset{\lambda}{\longrightarrow} B$ σ-weakly for some $B \in \mathcal{B}(\mathcal{H})$. Let $C, D \in \mathcal{A}$. Then it is trivial to see that $C^*A_\lambda D \underset{\lambda}{\longrightarrow} 0$ σ-weakly and $\pi(C^*)\pi(A_\lambda)\pi(D) \underset{\lambda}{\longrightarrow} \pi(C^*)B\pi(D)$ σ-weakly. Since, by Proposition 2.3.1, V is bounded on $\mathcal{H}$ we have

$$\varphi(C^*A_\lambda D) = V^*\pi(C^*A_\lambda D)V = V^*\pi(C^*)\pi(A_\lambda)\pi(D)V \underset{\lambda}{\longrightarrow} V^*\pi(C^*)B\pi(D)V$$

σ-weakly. Well, φ is σ-weakly - σ-weakly closable by Proposition 2.3.2 and so $V^*\pi(C^*)B\pi(D)V = 0$. Then, for any $x, y \in \mathcal{H}$

$$\langle \pi(C)Vx, B\pi(D)Vy\rangle = \langle x, V^*\pi(C^*)B\pi(D)Vy\rangle = 0,$$

and, since (π, V) is a minimal representation, $B = 0$. Therefore π is closable. □

In the application of the theorem of Kraus to the generators of uniformly continuous QMSs on $\mathcal{B}(\mathcal{H})$, π is a σ-weakly continuous unital *-representation so, for a cyclic vector $\omega \in \mathcal{K}$, the map $\mathcal{B}(\mathcal{H}) \ni A \mapsto \langle \omega, \pi(A)\omega \rangle$ is positive and σ-weakly continuous. Since we have a characterization of such maps, namely positive trace-class operators acting on $\mathcal{B}(\mathcal{H})$ via the trace duality, we can conclude this map has the form

$$\langle \omega, \pi(A)\omega \rangle = \sum_{n=1}^{\infty} \langle x_n, Ax_n \rangle$$

where $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$. Unfortunately, if we replace $\mathcal{B}(\mathcal{H})$ with a (not necessarily closed) *-subalgebra $\mathcal{A}$ and we only assume that the unital *-representation $\pi : \mathcal{A} \to \mathcal{K}$ is (σ-weakly, σ-weakly)-closable (which is guaranteed by Proposition 2.3.6) we do not know the form of the map $\mathcal{A} \ni A \mapsto \langle \omega, \pi(A)\omega \rangle$. This seems to be the missing ingredient in order to obtain an analogue result of Kraus for general QMS on $\mathcal{B}(\mathcal{H})$.

2.4 Examples

We will now proceed to look at three examples of QMSs where we verify that their generators satisfy the form given by Corollary 2.2.9. We identify the linear maps G, V, the representation π, the Hilbert space $\mathcal{K}$ and the linear subspace U of $\mathcal{H}$ as in Corollary 2.2.9. Moreover we prove that the subspace U is dense in $\mathcal{H}$ in the first two examples.

Example 2.4.1. ***(Heat Flow, Arveson [2])*** *Define $P = \frac{1}{i}\frac{d}{dx}$ and Q to be multiplication by x where P and Q act on $L_2(\mathbb{R})$. Further, for $A \in \mathcal{B}(L_2(\mathbb{R}))$ define*

$$D_P(A) = i(PA - AP) \quad \textit{and} \quad D_Q(A) = i(QA - AQ)$$

where D_P and D_Q are unbounded operators on $\mathcal{B}(L_2(\mathbb{R}))$. Next, define $L : D(L)(\subseteq \mathcal{B}(L_2(\mathbb{R})) \to \mathcal{B}(L_2(\mathbb{R}))$ by $L = D_P^2 + D_Q^2$. Then L generates a QMS.

The fact that L generates a QMS was proved by Arveson [2]. By expanding L, we have

$$L(A) = 2\left(PAP + QAQ\right) - \left(P^2 + Q^2\right)A - A\left(P^2 + Q^2\right)$$

for all $A \in D(L)$. Note here that this expression is in the form given by Corollary 2.2.9 with $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}$, $V = \sqrt{2}P \oplus Q$, $\pi(A) = A \oplus A$, and $G = -(P^2 + Q^2)$.

Let $e \in L_2(\mathbb{R})$ of norm one such that $|e\rangle\langle e| \in D(L)$, say $e(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-x^2/2\right)$ for example, let $T = |e\rangle\langle e|$ and $h = e$ be an associate vector for T. Since $Th = e$ we have

$$U = \{u \in L_2(\mathbb{R}) : |u\rangle\langle e| \in \mathcal{A}\}$$

where $\mathcal{A}$ is the domain algebra of L. Let

$$U' = \{u \in L_2(\mathbb{R}) : u', u'', Qu, Q^2u \in L_2(\mathbb{R})\}.$$

It is an easy exercise to check that $U' \subseteq U$ and, since the Schwartz class is norm dense in $L_2(\mathbb{R})$, we have that U is norm dense in $L_2(\mathbb{R})$.

Example 2.4.2. ***(Parthasarathy [32]-pg. 258)*** *Let $(B_t)_{t\geq 0}$ be a standard Brownian motion defined on the probability space $(\Omega, \mathcal{F}, P)$ and define $T_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by*

$$T_t A = \mathbb{E}\left[e^{iB_tV} A e^{-iB_tV}\right]$$

where V is a self-adjoint operator on $\mathcal{H}$. Then $(T_t)_{t\geq 0}$ is a QMS.

The fact that $(T_t)_{t\geq 0}$ is a QMS follows immediately from the Îto calculus but we can also see this without such tools. The fact that $T_0 = 1$ and $T_t(1) = 1$ are obvious. To prove $T_{t+s} = T_t T_s$ start with the identity

$$T_{t+s}A = \mathbb{E}\left[e^{i(B_{t+s}-B_s)V} e^{iB_sV} A e^{-iB_sV} e^{-i(B_{t+s}-B_s)V}\right]$$

and use the property of independent increments for Brownian motion to get the desired result. The remaining properties which qualify $(T_t)_{t\geq 0}$ as a QMS are fairly obvious. Now, suppose V is bounded. Let $(t_n)_{n\in\mathbb{N}} \subseteq [0,\infty)$ such that $t_n \to t$. Further, for $A \in \mathcal{B}(\mathcal{H})$,

$$\begin{aligned}\|\mathbb{E}\left[e^{iB_{t_n}V}Ae^{-iB_{t_n}V} - e^{iB_tV}Ae^{-iB_tV}\right]\| &\leq \int_\Omega \|e^{iB_{t_n}V}Ae^{-iB_{t_n}V} - e^{iB_tV}Ae^{-iB_tV}\|dP \\ &\leq \int_\Omega 2\|e^{i(B_{t_n}-B_t)V} - 1\|\|A\|dP \\ &\to 0\end{aligned}$$

by the Bounded Convergence Theorem since $B_{t_n}(\omega) \to B_t(\omega)$. So we have that $(T_t)_{t\geq 0}$ is a uniformly continuous QMS. Next, we claim that $T_tA = \mathbb{E}\left[e^{iB_t(adV)}A\right]$ where $(adV)A = VA - AV$ for all $A \in \mathcal{B}(\mathcal{H})$. To this end, it's an exercise to show that

$$(adV)^nA = \sum_{k=0}^{n}(-1)^k\binom{n}{k}V^{n-k}AV^k$$

which gives

$$\mathbb{E}\left[e^{iB_t(adV)}A\right] = T_tA.$$

Further,

$$T_tA = \frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\int_{\mathbb{R}} e^{-x^2/2}\frac{(ix\sqrt{t})^n}{(2n)!}(adV)^{2n}Adx.$$

Then, using our knowledge of Gaussian integrals, we'll find that

$$T_tA = e^{-\frac{1}{2}(adV)^2}A.$$

So the generator L of $(T_t)_{t\geq 0}$ is given by

$$L(A) = -\frac{1}{2}(adV)^2A = \frac{-1}{2}\left(V^2A + AV^2 - 2VAV\right).$$

Now, if V is unbounded then the generator is given "formally" by the above equation, that is, L can be realized as a sesquilinear form where

$$\langle u, L(A)v\rangle = \langle Vu, AVv\rangle + \langle u, -\frac{1}{2}V^2Av\rangle + \langle -\frac{1}{2}V^2A^*u, v\rangle.$$

Also, the generator has the form given in Corollary 2.2.9 with $G = -\frac{1}{2}V^2$. If $\mathcal{H} = L_2(\mathbb{R})$ and $V = i\frac{d}{dx}$ then let $e(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-x^2/2\right)$ and let $T = |e\rangle\langle e|$. Then

$h = e$ is an associate vector for T and it is an easy exercise to see that

$$U = \{u \in L_2(\mathbb{R}) : |u\rangle\langle e \in \mathcal{A}\} \supseteq \{f \in L_2(\mathbb{R}) : f', f'' \in L_2(\mathbb{R})\},$$

and therefore U is dense in $L_2(\mathbb{R})$.

Example 2.4.3. ***(Arveson [2] and similar examples produced in Fagnola [19] and Powers [36])*** *Let $\mathcal{H} = L_2[0,\infty)$ and define $U_t : \mathcal{H} \to \mathcal{H}$ by*

$$(U_t g)(x) = \begin{cases} g(x-t) & \text{if } x \geq t \\ 0 & \text{otherwise} \end{cases}$$

Then $(U_t)_{t\geq 0}$ is a strongly continuous semigroup of isometries whose generator $\mathfrak{D}$ is differentiation. Let $f \in L_2(0,\infty)$ be what we get by normalizing $u(x) = e^{-x}$ (i.e. $f = \frac{u}{\|u\|}$) then define $\omega : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ by $\omega(A) = \langle f, Af\rangle$. Define the completely positive maps $\phi_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ where

$$\phi_t(A) = \omega(A)E_t + U_t A U_t^*$$

for all $t \geq 0$ where E_t is the projection onto the subspace $L_2(0,t) \subseteq L_2(0,\infty)$. Then $(\phi_t)_{t\geq 0}$ is a QMS.

First note that for $A \in \mathcal{B}(\mathcal{H})$,

$$\omega(U_t A U_t^*) = \langle U_t^* f, A U_t^* f\rangle = \left\langle \frac{e^{-(\cdot+t)}}{\|u\|}, A\left(\frac{e^{-(\cdot+t)}}{\|u\|}\right)\right\rangle = e^{-2t}\langle f, Af\rangle = e^{-2t}\omega(A)$$

where the dots denote the variable of the function. We claim that $(\phi_t)_{t\geq 0}$ is a semi-group. First, we want to show that $\omega(\phi_t(A)) = \omega(A)$ for all $A \in \mathcal{B}(\mathcal{H})$. Indeed,

$$\begin{aligned}
\omega(\phi_t(A)) &= \omega(\omega(A)E_t + U_t A U_t^*) \\
&= \omega(A)\langle f, E_t f\rangle + \langle f, U_t A U_t^* f\rangle \\
&= \omega(A)\langle f, (1 - U_t U_t^*) f\rangle + e^{-2t}\omega(A) \qquad \text{since } E_t = 1 - U_t U_t^* \\
&= \omega(A)(1 - e^{-2t}) + e^{-2t}\omega(A) = \omega(A).
\end{aligned}$$

So we have that $\omega(\phi_t(A)) = \omega(A)$ for all $A \in \mathcal{B}(\mathcal{H})$. Next, we want to show $\phi_s\phi_t = \phi_{t+s}$. Let $A \in \mathcal{B}(\mathcal{H})$. Then

$$\phi_s\phi_t(A) = \omega(\phi_t(A))E_s + U_s(\omega(A)E_t + U_tAU_t^*)U_s^* = \omega(A)(E_s + U_sE_tU_s^*) + U_{s+t}A(U_{s+t})^*$$

and, since $E_{s+t} = E_s + U_sE_tU_s^*$, we have that

$$\phi_s\phi_t(A) = \omega(A)E_{s+t} + U_{s+t}A(U_{s+t})^* = \phi_{s+t}(A).$$

So we have that $(\phi_t)_{t\geq 0}$ is a QMS. If L denotes the generator of $(\phi_t)_{t\geq 0}$ and $D(L)$ denotes the domain of L then

$$\begin{aligned} L(A) &= \sigma - \text{weak} - \lim_{t\to 0} \frac{1}{t}(\phi_t(A) - A) \\ &= \sigma - \text{weak} - \lim_{t\to 0} \left(\frac{\omega(A)E_t}{t} + \frac{U_tAU_t^* - A}{t} \right) \quad \text{for all } A \in D(L). \end{aligned} \tag{2.11}$$

By Example 2.1.1 the generator of the QMS $(A \mapsto U_tAU_t^*)_{t\geq 0}$ is equal to $\alpha_{\mathfrak{D}}$ where $\alpha_{\mathfrak{D}}(A) = \mathfrak{D}A + A\mathfrak{D}^*$. For $A \in D(L) \cap D(\alpha_{\mathfrak{D}})$, if $u, v \in L_2[0,\infty)$ such that $u', v' \in L_2[0,\infty)$ and u and v are continuous at zero, then

$$\langle u, L(A)v\rangle = \omega(A)\overline{u(0)}v(0) + \langle u, \mathfrak{D}Av\rangle + \langle \mathfrak{D}A^*u, v\rangle$$

and one can check that the map $(A, u, v) \mapsto \omega(A)\overline{u(0)}v(0)$ is completely positive for appropriately chosen $A \in D(L)$ and $u, v \in L_2[0,\infty)$. By (2.11), if a bounded operator A belongs to $D(\alpha_{\mathfrak{D}})$ and the kernel of ω, (i.e. $\omega(A) = 0$), then $A \in D(L)$. Now fix a normalized vector $e \in L_2[0,\infty)$ such that $D(e) \in L_2[0,\infty)$ and $\langle e, f\rangle = 0$ (for example, take e to be what we get by normalizing $e_0 = g - \langle f, g\rangle f$ where g is any function in $L_2[0,\infty)$ such that $g' \in L_2[0,\infty)$ and $\|g\| = 1$) and use this vector e in place of Th to define the subspace U of Corollary 2.2.9, (for example take the positive finite rank operator T to be equal to $|e\rangle\langle e|$ and the associate vector h of T to be equal to e; it is easy to verify that $T \in D(L)$, $Th = e$, and $|Th\rangle\langle Th| \in D(L)$). Using the fact that $\ker\omega \cap D(\alpha_{\mathfrak{D}}) \subseteq D(L)$, is easy to verify that for all $x \in L_2[0,\infty)$

with $\mathfrak{D}(x) \in L_2[0,\infty)$ and $\langle x, f\rangle = 0$ we have that the following three conditions are satisfied: $|x\rangle\langle e| \in D(L)$, $|x\rangle\langle e|(|x\rangle\langle e|)^* = \|e\|^2|x\rangle\langle x| \in D(L)$, and $(|x\rangle\langle e|)^*|x\rangle\langle e| = \|x\|^2|e\rangle\langle e| \in D(L)$. Thus by Definition 2.2.5 $|x\rangle\langle x| \in \mathcal{A}$ where $\mathcal{A}$ denotes the domain algebra of L. Hence

$$\{x \in L_2[0,\infty) : \mathfrak{D}(x) \in L_2[0,\infty) \text{ and } \langle x, f\rangle = 0\} \subseteq U. \tag{2.12}$$

Arveson [2, the proposition of pg. 75] proves that the strong operator closure $\overline{\mathcal{A}}^{\text{SOT}}$ of the domain algebra is equal to the set of bounded operators A such that both A and its adjoint A^* have f as an eigenvector (necessarily corresponding to complex conjugate eigenvalues). Thus for $x \in L_2[0,\infty)$, if $A = |x\rangle\langle e| \in \mathcal{A}$ then $\langle f, e\rangle = 0$. Therefore

$$U \subseteq \{x \in L_2[0,\infty) : \langle x, f\rangle = 0\}. \tag{2.13}$$

We do not have a more precise description of U besides (2.12) and (2.13). Equation (2.13) shows that U is not dense in $\mathcal{H}$. Note that the domain algebra $\mathcal{A}$ contains operators which are not in the kernel of ω (since $|f\rangle\langle f| \in \overline{\mathcal{A}}^{\text{SOT}}$ by Arveson [2, the proposition of pg. 75]). Hence the operator V and the unital *-representation π which appear in the statement of Corollary 2.2.9 are non-zero. The operator G which appears in the statement of Corollary 2.2.9 is not necessarily equal to the generator $\mathfrak{D}$ of $(U_t)_{t\geq 0}$. Formulas for V, π and G are given in Corollary 2.2.9 and Theorems 2.2.6 and 2.2.8 and we do not know simpler formulas for this particular example.

Chapter 3

Closedness of the generator of a semigroup

The motivation for the results in this chapter were to find suitable conditions for the generator of a QMS to be closed with respect to different topologies to obtain some continuity results without such strong assumptions as in Section 2.3. The results we obtained were more general than we needed and so we will present them in their full generality.

3.1 Generators of $\sigma(X, F)$-continuous semigroups

In this section we introduce the notion of $\sigma(X, F)$-semigroups and we give sufficient conditions which imply that the generator of a $\sigma(X, F)$-semigroup acting on a Banach space X is $\sigma(X, F) - \sigma(X, F)$ closed. The main results of the section are Theorems 3.1.9 and 3.1.11.

Definition 3.1.1. *Let F be a subset of X^* which separates points in X (thus the topology $\sigma(X, F)$ on X is Hausdorff). We say that a semigroup $(T_t)_{t\geq 0}$ on X is a $\sigma(X, F)$**-semigroup** if T_t is $\sigma(X, F) - \sigma(X, F)$ continuous for all $t \geq 0$.*
*We say a semigroup $(T_t)_{t\geq 0}$ is $\sigma(X, F)$ **continous at zero** if for all $\eta \in F$ and for all $A \in X$ we have that if $t \searrow 0$ then $\eta(T_t A) \to \eta(A)$. We say that the semigroup is $\sigma(X, F)$ **continuous** if for all $\eta \in F$ and for all $A \in X$ we have that if $t \to s$ for some $s \geq 0$ (while t stays non-negative as well) then $\eta(T_t A) \to \eta(T_s A)$. A semigroup is said to be **exponentially bounded** if there exists $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T_t\| \leq Me^{\omega t}$ for all $t \geq 0$.*

Remark 3.1.2. *Let X be a normed space and F be a linear subspace of X^* which separates points in X. If $(T_t)_{t\geq 0}$ is a $\sigma(X,F)$-semigroup on X then the family of adjoint maps $(T_t^*)_{t\geq 0}$ with $T_t^* : F \to F$ is defined by $T_t^*(\eta) = \eta \circ T_t$ for all $\eta \in F$. It is easy to check that T_t^* is $\sigma(F,X) - \sigma(F,X)$ continuous for all $t \geq 0$. Further, if for any fixed $A \in X$, the map $t \mapsto T_tA$ is F-Dunford integrable then for fixed $\eta \in F$, the map $t \mapsto T_t^*(\eta)$ is X-Dunford integrable. Indeed, if the map $t \mapsto T_tA$ is F-Dunford integrable for each $A \in X$ then $\eta \circ T_tA \in L_1[0,\infty)$ for all $A \in X$ and $\eta \in F$, that is, $A \circ T_t^*\eta \in L_1[0,\infty)$ for all $A \in X$ and $\eta \in F$ but this is precisely what is needed for the map $t \mapsto T_t^*(\eta)$ to be X-Dunford integrable.*

Lemma 3.1.3. *Let X be a Banach space and suppose that F is a linear subspace of X^* which norms X. Let $(T_t)_{t\geq 0}$ be a semigroup on X which is $\sigma(X,F)$ continuous at zero. Then $(T_t)_{t\geq 0}$ is exponentially bounded.*

Proof. Let $A \in X$.

Claim: There exists $\delta_A > 0$ such that if $C_A = \{\|T_tA\| : 0 \leq t \leq \delta_A\}$ then $\sup C_A < \infty$.

If not, we can find a sequence $(t_n)_{n\in\mathbb{N}} \subset [0,\infty)$ such that $t_n \searrow 0$ but $\|T_{t_n}A\| \to \infty$. For $\eta \in F$ we have that $\eta(T_{t_n}A) \to \eta(A)$ so $\{|\eta(T_{t_n}A)| : n \in \mathbb{N}\}$ is bounded for all $\eta \in F$ so, by the uniform boundedness principle and the assumption that F_2 norms X, $\{\|T_{t_n}A\| : n \in \mathbb{N}\}$ is bounded contradicting the fact that $\|T_{t_n}A\| \to \infty$. So, let $\delta_A > 0$ such that $M = \sup C_A < \infty$.

We now claim that for all $A \in X$, $\sup\{\|T_tA\| : 0 \leq t \leq 1\} < \infty$. Indeed, for any $t \in [0,1]$, we have that $t = n\delta_A + \epsilon$ for some integer n such that $0 \leq n \leq \lfloor \delta_A^{-1} \rfloor$ and $0 \leq \epsilon < \delta_A$. Then

$$\|T_tA\| = \|T_{\delta_A}^n T_\epsilon A\| \leq \|T_{\delta_A}\|^n M \leq K$$

where $K = \max\{M, \|T_{\delta_A}\|^{\lfloor \delta_A^{-1} \rfloor} M\}$ so we have that $\sup\{\|T_tA\| : t \in [0,1]\} < \infty$.

By the uniform boundedness principle we obtain that $M = \sup\{\|T_t\| : t \in [0,1]\} < \infty$. Let $t \in [0,\infty)$. Then $t = n + \epsilon$ for some $n \in \mathbb{N}$ and $0 \leq \epsilon < 1$. Then

$$\|T_t\| = \|T_n T_\epsilon\| \leq \|T_1\|^n \|T_\epsilon\| = Me^{\omega n} \leq Me^{\omega t}$$

where $\omega = \ln \|T_1\|$. Therefore we have that $(T_t)_{t\geq 0}$ is exponentially bounded. □

A definition for the generator of a QMS was given in Chapter 1 but since we are now looking at more general types of semigroups, the limit we use to define the generator will be with respect to the subspace F of the dual space X^*. This change is made more specific in the following definition.

Definition 3.1.4. *Let X be a normed space, F be a subspace of X^* which separates points in X, and $(T_t)_{t\geq 0}$ be a semigroup on X which is $\sigma(X,F)$ continuous at 0. The **generator** of $(T_t)_{t\geq 0}$ is defined as the linear operator L on X, whose domain $D(L)$ consists of those $A \in X$ for which there exists an element $B \in X$ with the property that*

$$\eta(B) = \lim_{t\to 0} \frac{\eta(T_t A - A)}{t}, \quad \textit{for all } \eta \in F.$$

If $A \in D(L)$ then L is defined by $LA = B$.

Definition 3.1.5. *Let X be a normed space and F be a subspace of X^* which separates points in X. Let $(T_t)_{t\geq 0}$ be a semigroup on X and let L denote its generator. The semigroup $(T_t)_{t\geq 0}$ is called **F-integrable** if there exists an $\omega \in \mathbb{R}$ such that for every $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ we have that λ belongs to the resolvent of L and moreover the bounded linear operator $(\lambda - L)^{-1}$ is $\sigma(X,F) - \sigma(X,F)$ continuous.*

Remark 3.1.6. *Let X be a normed space and F be a subspace of X^* which separates points in X. Let $(T_t)_{t\geq 0}$ be an F-integrable semigroup on X, let L be its generator, and $D(L)$ be the domain of L. Then $L : D(L) \to X$ is $\sigma(X,F) - \sigma(X,F)$ closed.*

Proof. In order to show that L is $\sigma(X,F) - \sigma(X,F)$ closed we fix $\omega \in \mathbb{R}$ as in Definition 3.1.5 and $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ and we show that $\lambda - L : D(L) \to X$ is

$\sigma(X,F)$-$\sigma(X,F)$ closed. Indeed suppose $(A_\gamma)_\gamma \subseteq D(L)$ is a net which converges to some element $A \in X$ in the $\sigma(X,F)$ topology and $((\lambda - L)(A_\gamma))_\gamma$ converges in the $\sigma(X,F)$ topology to some element $B \in X$. Since $(\lambda - L)^{-1}$ is $\sigma(X,F) - \sigma(X,F)$ continuous we have that

$$(\lambda - L)^{-1}(\lambda - L)A_\gamma \to (\lambda - L)^{-1}B$$

i.e., $A_\gamma \to (\lambda - L)^{-1}B$ in the $\sigma(X,F)$ topology, and hence $(\lambda - L)^{-1}B = A$ so $A \in D(L)$ and $(\lambda - L)A = B$. Therefore $\lambda - L : D(L) \to X$ is $\sigma(X,F) - \sigma(X,F)$ closed and so $L : D(L) \to X$ is $\sigma(X,F) - \sigma(X,F)$ closed. □

Lemma 3.1.7. *Let X be a Banach space. Suppose that F is a norm-closed subspace of X^* which is a norming set for X and $(F, \sigma(F,X))$ and $(X, \sigma(X,F))$ are Mazur spaces. Further, suppose $(T_t)_{t\geq 0}$ is a $\sigma(X,F)$-semigroup on X which is exponentially bounded and for all $A \in X$, the map $t \mapsto T_tA$ is F-measurable with respect to the Borel σ-algebra. Then there exists $\omega \in \mathbb{R}$ such that for all $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ the map $[0,\infty) \ni t \mapsto e^{-\lambda t}T_tA$ is F-Pettis integrable for all $A \in X$ and if $R(\lambda) : X \to X$ is defined by*

$$R(\lambda)A = (P) - \int_0^\infty e^{-\lambda t}T_tA\,dt, \quad \textit{for all } A \in X \tag{3.1}$$

then $R(\lambda)$ is $\sigma(X,F) - \sigma(X,F)$ continuous.

Proof. Let real numbers M and ω be such that $\|T_t\| \leq Me^{\omega t}$ for all $t \geq 0$. Let $\lambda \in \mathbb{C}$ such that $\Re\lambda > \omega$. Since for all $A \in X$, $t \mapsto T_tA$ is F-measurable, $(F, \sigma(F,X))$ is a Mazur space, and $\|T_tA\| \leq \|A\|Me^{\omega t} \in L_1([0,\infty), d\mu_\lambda)$ where $d\mu_\lambda(t) = e^{-\lambda t}dt$, we obtain, by Theorem 1.2.9, that for all $A \in X$ the function $t \mapsto T_tA$ is an X-valued F-Pettis integrable function with respect to the measure μ_λ. Thus for all $A \in X$, there exists $x_A \in X$ such that

$$\eta(x_A) = \int_0^\infty \eta(T_tA)d\mu_\lambda(t), \quad \text{for all } \eta \in F. \tag{3.2}$$

Define $R(\lambda) : X \to X$ by $R(\lambda)A = x_A$. In order to show that $R(\lambda)$ is bounded notice that for $A \in X$,

$$\|R(\lambda)A\| = \|x_A\| = \sup_{\substack{\eta\in F\\ \|\eta\|=1}} |\eta(x_A)| \quad \text{since } F \text{ norms X}$$
$$= \sup_{\substack{\eta\in F\\ \|\eta\|=1}} \left|\int_0^\infty \eta(T_tA)d\mu_\lambda(t)\right| \leq \sup_{\substack{\eta\in F\\ \|\eta\|=1}} \int_0^\infty |\eta(T_tA)|\, e^{-\lambda t}dt \leq \|A\| \int_0^\infty Me^{(\omega-\lambda)t}dt.$$

So $R(\lambda)$ is bounded. Lastly, in order to show that $R(\lambda)$ is $\sigma(X,F) - \sigma(X,F)$ continuous we need to show that $\eta \circ R(\lambda) \in F$ for all $\eta \in F$. By (3.2) we have that for $\eta \in F$,

$$(\eta \circ R(\lambda))(A) = \int_0^\infty \eta(T_tA)d\mu_\lambda(t) = \int_0^\infty T_t^*(\eta)(A)d\mu_\lambda(t). \tag{3.3}$$

Since F is a Banach space, $(X, \sigma(X,F))$ is a Mazur space, $t \mapsto T_t^*(\eta)$ is X-measurable by Remark 3.1.2 and bounded by an $L_1([0,\infty), \mu_\lambda)$ function, we have by Theorem 1.2.9 that $t \mapsto T_t^*\eta$ is an F-valued X-Pettis integrable function with respect to the measure μ_λ for all $\eta \in F$. Thus for all $\eta \in F$ there exists $\psi_\eta \in F$ so that

$$A(\psi_\eta) = \int_0^\infty A(T_t^*(\eta))d\mu_\lambda(t), \quad \text{for all } A \in X.$$

Combining with (3.3) we obtain

$$\psi_\eta(A) = \int_0^\infty \eta(T_tA)d\mu_\lambda(t) = (\eta \circ R(\lambda)(A), \quad \text{for all } A \in X$$

and so $\eta \circ R(\lambda) = \psi_\eta \in F$. So we have that the semigroup $(T_t)_{t\geq 0}$ is $\sigma(X,F)$-integrable. □

The conclusion of Lemma 3.1.7 states that the Laplace transform of the semigroup is a bounded operator which is continuous with respect to the same topologies as each member of the semigroup family. This conclusion becomes an assumption of the next lemma.

Lemma 3.1.8. *Let X be a Banach space and let F be a linear subspace of X^* which is norming for X. Assume that $(T_t)_{t\geq 0}$ is a $\sigma(X,F)$-semigroup on X such that there* *exists $\omega \in \mathbb{R}$ such that for all $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ the map $[0,\infty) \ni t \mapsto e^{-\lambda t}T_t A$ is F-Pettis integrable for all $A \in X$. Define*

$$R(\lambda)A = (P) - \int_0^\infty e^{-\lambda t} T_t A dt, \quad \textit{for all } A \in X.$$

Then $R(\lambda) = (\lambda - L)^{-1}$ for all such λ.

Proof. Let $\eta \in F$, $A \in X$, and $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$. Since $R(\lambda)A = \int_0^\infty e^{-\lambda t} T_t A dt$ for all $A \in X$, we have that

$$\eta(R(\lambda)A) = \int_0^\infty e^{-\lambda t} \eta(T_t A) dt, \quad \text{for all } A \in X, \quad \eta \in F.$$

Since T_t is $\sigma(X,F)$-$\sigma(X,F)$ continuous and F is a linear subspace of X^* which is norming for X, by Proposition 1.2.12, we have that $R(\lambda)T_t A = T_t R(\lambda)A$. Thus,

$$\begin{aligned}
\eta(t^{-1}(T_t - I)R(\lambda)A) &= \eta(t^{-1}R(\lambda)(T_t A) - t^{-1}R(\lambda)A) \quad \text{since } R(\lambda)T_t A = T_t R(\lambda)A \\
&= \frac{1}{t}\int_0^\infty e^{-\lambda s}\eta(T_s T_t A) ds - \frac{1}{t}\int_0^\infty e^{-\lambda s}\eta(T_s A) ds \\
&= \frac{1}{t}\int_t^\infty e^{-\lambda(u-t)}\eta(T_u A) du - \frac{1}{t}\int_0^\infty e^{-\lambda s}\eta(T_s A) ds \\
&= \frac{1}{t}\int_0^\infty (e^{-\lambda(u-t)} - e^{-\lambda u})\eta(T_u A) du - \frac{1}{t}\int_0^t e^{-\lambda(u-t)}\eta(T_u A) du.
\end{aligned}$$

Further,

$$\frac{1}{t}\int_0^\infty (e^{-\lambda(u-t)} - e^{-\lambda u})\eta(T_u A) du \to \lambda\eta(R(\lambda)A)$$

and, by Lebesgue's Differentiation Theorem,

$$\frac{1}{t}\int_0^t e^{-\lambda(u-t)}\eta(T_u A) du \to \eta(A).$$

So, $R(\lambda)A \in \mathcal{D}(L)$ for all $A \in X$ and $\eta((\lambda - L)R(\lambda)A) = \eta(A)$ for all $A \in X$ and all $\eta \in F$. Since F is norming for X we obtain that $(\lambda - L)R(\lambda)A = A$ for all $A \in X$. Since T_t and $R(\lambda)$ commute, we obtain similarly $R(\lambda)(\lambda - L)A = A$ for all $A \in D(L)$. Thus $R(\lambda) = (\lambda - L)^{-1}$. □

Theorem 3.1.9. *Let X be a Banach space and let F be a norm closed subspace of X^* which is a norming set for X. Assume that $(F, \sigma(F, X))$ as well as $(X, \sigma(X, F))$ are Mazur spaces. Let $(T_t)_{t\geq 0}$ be a $\sigma(X, F)$-semigroup which is $\sigma(X, F)$ continuous at zero. Then the generator, L, of $(T_t)_{t\geq 0}$ is $\sigma(X, F) - \sigma(X, F)$ closed.*

Proof. By Lemma 3.1.3, $(T_t)_{t\geq 0}$ is exponentially bounded.

We claim that $(T_t)_{t\geq 0}$ is $\sigma(X, F)$-integrable. In order to prove that, fix $A \in X$ and $\eta \in F$ and notice that if $t \searrow s$ we have

$$|\eta(T_t A) - \eta(T_s(A))| = |\eta(T_{t-s} T_s A - T_s A)| \to 0$$

since $(T_t)_{t\geq 0}$ is $\sigma(X, F)$ continuous at zero. So $t \mapsto \eta(T_t A)$ is right-continuous and so Borel measurable. Hence, by Lemma 3.1.7, there exists $\omega \in \mathbb{R}$ such that for all $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ there exists a bounded linear operator $R(\lambda) : X \to X$ which is $\sigma(X, F) - \sigma(X, F)$ continuous and satisfies Equation (3.1). Therefore by Lemma 3.1.8 we obtain that $R(\lambda) = (\lambda - L)^{-1}$ for all such λ. Thus all $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ belong to the resolvent of L and for all such λ we have that $(\lambda - L)^{-1}$ is $\sigma(X, F) - \sigma(X, F)$ continuous. Thus $(T_t)_{t\geq 0}$ is $\sigma(X, F)$-integrable. Hence by Remark 3.1.6 we obtain that L is $\sigma(X, F) - \sigma(X, F)$ closed.

□

We also have an analogous result when we replace the property of Mazur with the Krein-Smulian property. This result strengthens Proposition 3.1.4 of [4]. First, we need a lemma.

Lemma 3.1.10. *Let X be a Banach space and F be a linear (not necessarily closed) subspace of X^* which is norming for X. Assume that (X, F) and (F, X) satisfy the Krein-Smulian property. Let $(T_t)_{t\geq 0}$ be an $\sigma(X, F)$-semigroup on X which is $\sigma(X, F)$ continuous. Then there exists $\omega \in \mathbb{R}$ such that for all $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ the map*

$[0,\infty) \ni t \mapsto e^{-\lambda t} T_t A$ is F-Pettis integrable for all $A \in X$ and if we define

$$R(\lambda)A = (P) - \int_0^\infty e^{-\lambda t} T_t A dt, \quad \textit{for all } A \in X$$

then $R(\lambda)$ is $\sigma(X, \overline{F})$-$\sigma(X, F)$ continuous for all such λ where $\overline{F}$ is the closure of F inside X^*.

Proof. First, by Lemma 3.1.3, we obtain that $(T_t)_{t \geq 0}$ is exponentially bounded. Thus there exists $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T_t\| \leq M e^{\omega t}$ for all $t \geq 0$. Then we fix $\lambda \in \mathbb{C}$ with $\Re\lambda > \omega$ and $A \in X$. Notice that the function $[0,\infty) \ni t \mapsto e^{-\lambda t} T_t A$ is an X-valued F-Pettis integrable function with respect to Lebesgue measure. Indeed we have that $e^{-\lambda t}\|T_t A\| \leq \|A\| M e^{(\lambda - \omega)t} \in L_1([0,\infty), dt)$. Also, since $t \mapsto T_t A$ is $\sigma(X, F)$ continuous, it is F-measurable with respect to the Borel σ-algebra of $[0,\infty)$. Finally (X, F) satisfies the Krein-Smulian property. Thus by Theorem 1.2.7, $t \mapsto e^{-\lambda t} T_t A$ is an F-Pettis integrable function with respect to the Lebesgue measure. So, there exists $x_A \in X$ such that

$$\eta(x_A) = \int_0^\infty e^{-\lambda t} \eta(T_t A) dt, \qquad \text{for all } \eta \in F.$$

Define $R(\lambda) : X \to X$ by $R(\lambda)A = x_A$.

Next notice that $R(\lambda) : X \to X$ is bounded since for $A \in X$ we have

$$\|R(\lambda)A\| = \|x_A\| = \sup_{\substack{\eta \in F \\ \|\eta\|=1}} |\eta(x_A)| = \sup_{\substack{\eta \in F \\ \|\eta\|=1}} \left| \int_0^\infty e^{-\lambda t} \eta(T_t A) dt \right| \leq \|A\| \int_0^\infty M e^{(\omega - \lambda)t} dt.$$

Next, we show that $R(\lambda)$ is $\sigma(X, \overline{F}) - \sigma(X, F)$ continuous where $\overline{F_1}$ is the closure of F_1 inside X^*. Fix $\eta \in F$. We need to show that $\eta \circ R(\lambda) \in \overline{F}$. First, notice that

$$(\eta \circ R(\lambda))(A) = \int_0^\infty e^{-\lambda t} \eta(T_t A) dt = \int_0^\infty e^{-\lambda t} T_t^*(\eta)(A) dt, \qquad \text{for all } A \in X. \tag{3.4}$$

Define $f : [0,\infty) \to F$ by $f(t) = e^{-\lambda t} T_t^*(\eta)$. Observe that f takes values in F since F is a linear subspace of X^* and $(T_t)_{t \geq 0}$ is a $\sigma(X, F)$-semigroup. Since (F, X) has the

Krein-Smulian property we obtain by Theorem 1.2.7 that f is an X-Pettis integrable function with integrals in $\overline{F}$ i.e. there exists $\gamma \in \overline{F}$ so that

$$A(\gamma) = \int_0^\infty e^{-\lambda t} A(\eta \circ T_t) dt, \quad \text{for all } A \in X.$$

That is,

$$\gamma(A) = \int_0^\infty e^{-\lambda t} \eta(T_t A) dt = \eta \circ R(\lambda)(A), \quad \text{for all } A \in X.$$

Hence $\eta \circ R(\lambda) = \gamma \in \overline{F}$ and therefore $R(\lambda)$ is $\sigma(X, \overline{F}) - \sigma(X, F)$ continuous. □

Theorem 3.1.11. *Let X be a Banach space and F be a linear (not necessarily closed) subspace of X^* which is norming for X. Assume that (X, F) and (F, X) satisfy the Krein-Smulian property. Let $(T_t)_{t \geq 0}$ be an $\sigma(X, F)$-semigroup on X which is $\sigma(X, F)$-continuous. Then the generator $L : D(L) \to X$ of the semigroup $(T_t)_{t \geq 0}$ is $\sigma(X, F) - \sigma(X, \overline{F})$ closed where $\overline{F}$ is the closure of F inside X^*.*

Proof. First, by Lemma 3.1.3, we obtain that $(T_t)_{t \geq 0}$ is exponentially bounded. Thus there exists $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T_t\| \leq M e^{\omega t}$ for all $t \geq 0$. By Lemma 3.1.10, the map $[0, \infty) \ni t \mapsto e^{-\lambda t} T_t A$ is F-Pettis integrable for all $A \in X$ and if we define

$$R(\lambda) A = (P) - \int_0^\infty e^{-\lambda t} T_t A dt, \quad \text{for all } A \in X$$

then $R(\lambda)$ is $\sigma(X, \overline{F})$-$\sigma(X, F)$ continuous. By Lemma 3.1.8 we obtain that $R(\lambda) = (\lambda - L)^{-1}$. Thus all $\lambda \in \mathbb{C}$ with $\Re \lambda > \omega$ belong to the resolvent of L and for all such λ we have that $(\lambda - L)^{-1}$ is $\sigma(X, \overline{F}) - \sigma(X, F)$ continuous. Thus $(T_t)_{t \geq 0}$ is $\sigma(X, F)$-integrable. Hence by Remark 3.1.6 we obtain that $L : D(L) \to X$, is $\sigma(X, F) - \sigma(X, \overline{F})$ closed. □

3.2 Applications to quantum Markov semigroups

In this section we apply Theorem 3.1.11 to quantum Markov semigroups $(T_t)_{t \geq 0}$ where each T_t is WOT-WOT continuous, that is, $\sigma(\mathcal{B}(\mathcal{H}), F)$ continuous where F is the space of finite rank operators on $\mathcal{H}$. Then we look at two examples of QMSs and verify that each T_t is WOT-WOT continuous.

Corollary 3.2.1. *Let $\mathcal{H}$ be a Hilbert space and $(T_t)_{t\geq 0}$ be a WOT-semigroup on $\mathcal{B}(\mathcal{H})$ which is WOT continuous at 0. Then its generator is WOT-σ-weakly closed. In particular, the generator of any quantum Markov WOT-semigroup is WOT-σ-weakly closed.*

Proof. Let X be $\mathcal{B}(\mathcal{H})$ and F be the space of finite rank operators as a linear subspace of $\mathcal{B}(\mathcal{H})^*$. Then (X, F) satisfies the Krein-Smulian Property since the $\sigma(X, F)$ topology is the WOT topology on $\mathcal{B}(\mathcal{H})$ and the unit ball of $\mathcal{B}(\mathcal{H})$ is WOT compact. Also (F, X) satisfies the Krein-Smulian Property by Alaoglu's Theorem. Thus we can simply apply Theorem 3.1.11 to obtain that the generator $L : D(L) \to \mathcal{B}(\mathcal{H})$, is $\sigma(\mathcal{B}(\mathcal{H}), F) - \sigma(\mathcal{B}(\mathcal{H}), \overline{F})$ closed. This gives the desired result since the $\sigma(\mathcal{B}(\mathcal{H}), F)$ topology is the WOT topology and the $\sigma(\mathcal{B}(\mathcal{H}), \overline{F})$ topology is the σ-weak topology (since the closure of the linear space of finite rank operators inside $\mathcal{B}(\mathcal{H})^*$ is equal to the closure of finite rank operators inside the predual of $\mathcal{B}(\mathcal{H})$). □

The assumption that the quantum Markov semigroup be a WOT-semigroup is true for many examples. We will look at two of them now. The first is a basic example which gives a way to produce a QMS from a classical semigroup.

Example 3.2.2. *Let $(V_t)_{t\geq 0}$ be a strongly continuous contraction semigroup on a Hilbert space $\mathcal{H}$. Define $T_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by*

$$T_t A = V_t A V_t^* \quad , \quad \textit{for all } A \in \mathcal{B}(\mathcal{H}).$$

It is easy to see that $(T_t)_{t\geq 0}$ is a QMS. To show that for a fixed $t \geq 0$, T_t is WOT-WOT continuous, let $(A_\lambda)_\lambda \subseteq \mathcal{B}(\mathcal{H})$ such that $A_\lambda \xrightarrow{WOT} A$ for some $A \in \mathcal{B}(\mathcal{H})$. Then, for any $x, y \in \mathcal{H}$,

$$\langle x, T_t A_\lambda y\rangle = \langle x, V_t A_\lambda V_t^* y\rangle = \langle V_t^* x, A_\lambda V_t^* y\rangle \to \langle V_t^* x, A V_t^* y\rangle = \langle x, T_t A y\rangle.$$

Therefore $(T_t)_{t\geq 0}$ is a WOT-semigroup.

The next example is given in [2] and similar examples where produced in [19] and [36].

Example 3.2.3. *Let $\mathcal{H} = L_2[0,\infty)$ and define $V_t : \mathcal{H} \to \mathcal{H}$ by*

$$(V_t g)(x) = \begin{cases} g(x-t) & \text{if } x \geq t \\ 0 & \text{otherwise} \end{cases}$$

Then $(V_t)_{t\geq 0}$ is a strongly continuous semigroup of isometries whose generator is the differentiation operator. Let $f \in L_2[0,\infty)$ be what we get by normalizing $u(x) = e^{-x}$ (i.e. $f = \frac{u}{\|u\|}$). Then define $\omega : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ by $\omega(A) = \langle f, Af\rangle$. Define the completely positive maps $T_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ where

$$T_t(A) = \omega(A)E_t + V_t A V_t^*$$

for all $t \geq 0$ where E_t is the projection onto the subspace $L_2[0,t) \subseteq L_2[0,\infty)$. Then $(T_t)_{t\geq 0}$ is a QMS.

Let $t \geq 0$. To see that T_t is WOT-WOT continuous, let $(A_\lambda)_\lambda \subseteq \mathcal{B}(\mathcal{H})$ such that $A_\lambda \xrightarrow{WOT} A$ for some $A \in \mathcal{B}(\mathcal{H})$. Then, for any $h, g \in \mathcal{H}$,

$$\begin{aligned} \langle h, T_t(A_\lambda)g\rangle &= \langle h, \omega(A_\lambda)E_t g\rangle + \langle h, V_t A_\lambda V_t^* g\rangle \\ &= \langle f, A_\lambda f\rangle\langle h, E_t g\rangle + \langle V_t^* h, A_\lambda V_t^* g\rangle \\ &\to \langle f, Af\rangle\langle h, E_t g\rangle + \langle V_t^* h, A V_t^* g\rangle \\ &= \langle h, T_t(A)g\rangle. \end{aligned}$$

Therefore T_t is WOT-WOT continuous.

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