# AN EXAMPLE OF A NON-SEPARABLE HILBERT SPACE

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Let $C(\mathbb{R})$ be the set of complex valued functions on $\mathbb{R}$ and define $\|\cdot\| : C(\mathbb{R}) \to [0,\infty]$ by

$$\|f\|_0 = \left( \lim_{N\to\infty} \frac{1}{N} \int_{-N}^{N} |f(x)|^2 dx \right)^{1/2}$$

and let $\mathcal{H}_0 = \{f \in C(\mathbb{R}) : \|f\|_0 < \infty\}$. Despite the notation, $\|\cdot\|_0$ is not a norm on $\mathcal{H}_0$. In fact, for any $f \in L_2(\mathbb{R})$ we have that $\|f\|_0 = 0$. We do, however, have that $\|\cdot\|_0$ is a semi-norm on $\mathcal{H}_0$ so, if we let $I = \{f \in \mathcal{H}_0 : \|f\|_0 = 0\}$, then $I$ is a closed subspace of $\mathcal{H}_0$ and so $\mathcal{H}_1 = \mathcal{H}_0/I$ is a normed vector space with norm $\|\cdot\|$ defined by

$$\|f + I\| = \inf_{\varphi\in I} \|f - \varphi\|_0$$

for all $f + I \in \mathcal{H}_1$. Note, by how we defined $I$, that for any $f \in \mathcal{H}_0$ we have $\|f + I\| = \|f\|_0$. Now define $\langle\cdot,\cdot\rangle : \mathcal{H}_1 \times \mathcal{H}_1 \to \mathbb{C}$ by

$$\langle f + I, g + I\rangle = \lim_{N\to\infty} \frac{1}{N} \int_{-N}^{N} f(x)\overline{g(x)}dx$$

for all $f + I, g + I \in \mathcal{H}_1$ to make $\mathcal{H}_1$ an inner product space. Then let $\mathcal{H}$ be the completion of $\mathcal{H}_1$ and so $\mathcal{H}$ is a Hilbert space containing $\mathcal{H}_1$. (The reader may be curious as to how we take an existential completion such as this. If we let $j : \mathcal{H}_1 \to \mathcal{H}_1^{**}$ be the natural map of $\mathcal{H}_1$ into its bidual, i.e., for $x \in \mathcal{H}_1$, $j(x) : \mathcal{H}_1^* \to \mathbb{C}$ where $j(x)(\phi) = \phi(x)$ for all $\phi \in \mathcal{H}_1^*$, then $j$ is a linear isometric embedding of $\mathcal{H}_1$ into $\mathcal{H}_1^{**}$ and so we take the closure of $\mathcal{H}_1$ in $\mathcal{H}_1^{**}$ to get $\mathcal{H}$. Further, since $\mathcal{H}_1^{**}$ is complete, we then have that $\mathcal{H}$ is complete.) Now we have our Hilbert space $\mathcal{H}$. Why is it non-separable? Consider $\mathcal{B} = \{\sin(\alpha x) : \alpha \in \mathbb{R}\}$. It is easy to check that $\mathcal{B}$ is an uncountable (clearly) orthonormal set. First, if we suppose $\alpha, \beta \in \mathbb{R}$ such that $\alpha \neq \beta$ then

$$\begin{aligned}
|\langle \sin(\alpha x), \sin(\beta x)\rangle| &= \left| \lim_{N\to\infty} \frac{1}{N} \int_{-N}^{N} \sin(\alpha x)\sin(\beta x)dx \right| \\
&= \left| \lim_{N\to\infty} \frac{1}{2N} \int_{-N}^{N} \cos((\alpha-\beta)x) - \cos((\alpha+\beta)x)dx \right| \\
&= \lim_{N\to\infty} \left| \frac{1}{2N} \frac{\sin((\alpha-\beta)x)}{\alpha-\beta}\bigg|_{-N}^{N} - \frac{\sin((\alpha+\beta)x)}{\alpha+\beta}\bigg|_{-N}^{N} \right| \\
&\leq \lim_{N\to\infty} \frac{1}{2N} \left( \frac{2}{\alpha-\beta} + \frac{2}{\alpha+\beta} \right) \\
&= 0
\end{aligned}$$

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Further, for $\alpha = \beta$ we have

$$\begin{aligned}
\langle \sin(\alpha x), \sin(\alpha x) \rangle &= \lim_{N\to\infty} \frac{1}{N} \int_{-N}^{N} \sin^2(\alpha x) dx \\
&= \lim_{N\to\infty} \frac{1}{2N} \int_{-N}^{N} 1 - \cos(2\alpha x) dx \\
&= \lim_{N\to\infty} \left(1 - \frac{\sin(2\alpha N)}{2\alpha N}\right) \\
&= 1
\end{aligned}$$

So, we have that $\mathcal{B}$ is an uncountable orthonormal set in $\mathcal{H}$ and so $\mathcal{H}$ cannot have a countable basis, hence $\mathcal{H}$ is non-separable.