A NOTE ON THE DEFINITION OF AN UNBOUNDED NORMAL OPERATOR

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1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and let $T : D(T)(\subseteq \mathcal{H}) \to \mathcal{H}$ be a generally unbounded linear operator. In order to begin the discussion about whether or not the operator T is normal, we need to require that T have a dense domain. This ensures that the adjoint operator T^* is defined.

In the literature, one sees two definitions for what it means for a densely defined operator T to be normal. The first requires that T be closed and that $T^*T = TT^*$ while the second requires that $D(T) = D(T^*)$ and $\|Tx\| = \|T^*x\|$ for all $x \in D(T) = D(T^*)$. There is no harm in having these competing definitions as they are, in fact, equivalent. While the equivalence of these statements seems to be widely known, the proof is not obvious, and my efforts to find a detailed and self-contained proof of this fact have left me empty-handed. I attempt to give such a proof here. I also want to note that while we say a *bounded* linear operator T is normal if and only if $T^*T = TT^*$, the lone requirement of $T^*T = TT^*$ for an unbounded densely defined operator does not, in general, imply T is normal. I used [1] and [2] to write this note.

2. Proof of the equivalence

We begin with a definition.

Definition 2.1. Let $T : D(T)(\subseteq \mathcal{H}) \to \mathcal{H}$ be a densely defined operator. We define the T**-inner product** on $D(T) \times D(T)$, denoted as $\langle \cdot, \cdot \rangle_T$, by the equation

$$\langle x, y \rangle_T = \langle x, y \rangle + \langle Tx, Ty \rangle, \quad \text{for all } x, y \in D(T).$$

It is easy to see that the T-inner product is, in fact, an inner product on $D(T)$. Further, we denote the T**-norm** on $D(T)$ as $\|\cdot\|_T$ and it is given by $\|x\|_T = \langle x, x \rangle_T^{1/2}$. Note that

$$\|x\|_T = \langle x, x \rangle_T^{1/2} = (\langle x, x \rangle + \langle Tx, Tx \rangle)^{1/2} = \left(\|x\|^2 + \|Tx\|^2\right)^{1/2}.$$

The following lemma is fairly elementary and widely known, but for the sake of having the notes self-contained we give the statement and proof.

Lemma 2.2. *Let $T : D(T)(\subseteq \mathcal{H}) \to \mathcal{H}$ be a densely defined operator. The inner product space $(D(T), \|\cdot\|_T)$ is a Hilbert space if and only if T is closed.*

Proof. First, assume T is closed. Let $(x_n)_{n\geq 1} \subseteq D(T)$ be a Cauchy sequence with respect to the T-norm. Then, for $n, m \in \mathbb{N}$, we have that

$$\|x_n - x_m\|^2 + \|Tx_n - Tx_m\|^2 = \|x_n - x_n\|_T^2 \to 0 \quad \text{as } m, n \to \infty$$

and so $(x_n)_{n\geq 1}$ and $(Tx_n)_{n\geq 1}$ are Cauchy with respect to the norm $\|\cdot\|$. So, there exists $x \in \mathcal{H}$ and $y \in \mathcal{H}$ such that $x_n \xrightarrow{\|\cdot\|} x$ and $Tx_n \xrightarrow{\|\cdot\|} y$. Since T is closed, $x \in D(T)$ and $Tx_n \to Tx$. Further,

$$\|x_n - x_m\|_T = \left(\|x_n - x\|^2 + \|Tx_n - Tx\|^2\right)^{1/2} \to 0$$

and so $x_n \xrightarrow{\|\cdot\|_T} x$ and therefore $(D(T), \|\cdot\|_T)$ is a Hilbert space.

Now, suppose $(D(T), \|\cdot\|_T)$ is a Hilbert space. Let $(x_n)_{n\geq 1} \subseteq D(T)$ such that $x_n \xrightarrow{\|\cdot\|} x \in \mathcal{H}$ and $Tx_n \xrightarrow{\|\cdot\|} y \in \mathcal{H}$. If $n, m \in \mathbb{N}$ then

$$\|x_n - x_m\|_T = \left(\|x_n - x_m\|^2 + \|Tx_n - Tx_m\|^2\right)^{1/2} \to 0 \quad \text{as } n, m \to \infty$$

and so $(x_n)_{n\geq 1}$ is Cauchy with respect to $\|\cdot\|_T$. Since $(D(T), \|\cdot\|_T)$ is complete, there exists $z \in D(T)$ such that $x_n \xrightarrow{\|\cdot\|_T} z$. Then

$$\|x_n - z\|^2 + \|Tx_n - Tz\|^2 = \|x_n - z\|_T^2 \to 0$$

and so $x_n \xrightarrow{\|\cdot\|} z$ and $Tx_n \xrightarrow{\|\cdot\|} Tz$. Hence $x = z \in D(T)$ and $Tx = Tz = y$. Therefore T is closed. $\square$

We are now ready to prove the equivalence of the two definitions.

Theorem 2.3. *Let $T : D(T)(\subseteq \mathcal{H}) \to \mathcal{H}$ be a densely defined operator. The following are equivalent:*

(i) T is closed and $T^*T = TT^*$.
(ii) $D(T) = D(T^*)$ and $\|Tx\| = \|T^*x\|$ for all $x \in D(T) = D(T^*)$.

(ii) $\Rightarrow$ (i). . Suppose $D(T) = D(T^*)$ and $\|Tx\| = \|T^*x\|$ for all $x \in D(T) = D(T^*)$. We first want to show that T is closed. To this end, let $(x_n)_{n\geq 1} \subseteq D(T)$ such that $x_n \xrightarrow{\|\cdot\|} x \in \mathcal{H}$

and $Tx_n \xrightarrow{\|\cdot\|} y \in \mathcal{H}$. Let $n, m \in \mathbb{N}$. Then

$$\|T^*x_n - T^*x_m\| = \|Tx_n - Tx_m\| \to 0 \quad \text{as } n, m \to \infty$$

since $(Tx_n)_{n\geq 1}$ is Cauchy. So $(T^*x_n)_{n\geq 1}$ is Cauchy and, since T^* is closed, we have that $x \in D(T^*) = D(T)$ and $T^*x_n \xrightarrow{\|\cdot\|} T^*x$. Then

$$\|Tx_n - Tx\| = \|T^*x_n - T^*x\| \to 0$$

and so $Tx_n \xrightarrow{\|\cdot\|} Tx$ but $Tx_n \to y$. Therefore $Tx = y$ and so T is closed.

Let $x, y \in D(T) = D(T^*)$. By the polarization identity

$$\langle Tx, Ty\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k\|Tx + i^kTy\|^2 = \frac{1}{4}\sum_{k=0}^{3} i^k\|T^*x + i^kT^*y\|^2 = \langle T^*x, T^*y\rangle. \tag{1}$$

Now, we would like to show $D(T^*T) = D(TT^*)$. To this end, let $x \in D(T^*T)$. Then $x \in D(T) = D(T^*)$ and $Tx \in D(T^*)$. So there exists $C > 0$ such that

$$|\langle Tx, Ty\rangle| \leq C\|y\|, \quad \text{for all } y \in D(T).$$

Then Equation (1) imples

$$|\langle T^*x, T^*y\rangle| \leq C\|y\|, \quad \text{for all } y \in D(T) = D(T^*).$$

Thus $T^*x \in D(T^{**})$. Since T is closed, we have that $T = T^{**}$ and so $T^*x \in D(T)$. Hence $x \in D(TT^*)$ and so $D(T^*T) \subseteq D(TT^*)$. Similarly, if $x \in D(TT^*)$ then $x \in D(T^*) = D(T)$ and $T^*x \in D(T) = D(T^{**})$. So there exists $C > 0$ such that

$$|\langle T^*x, T^*y\rangle| \leq C\|y\|, \quad \text{for all } y \in D(T) = D(T^*)$$

and so, by Equation (1), we have that

$$|\langle Tx, Ty\rangle| \leq C\|y\|, \quad \text{for all } y \in D(T) = D(T^*)$$

and thus $Tx \in D(T^*)$. Therefore $x \in D(T^*T)$ and so $D(TT^*) = D(T^*T)$. Now, let $x \in D(T^*T) = D(TT^*)$ and let $y \in D(T) = D(T^*)$. Then, by Equation (1), we have that $\langle Tx, Ty\rangle = \langle T^*x, T^*y\rangle$, that is, $\langle T^*Tx, y\rangle = \langle TT^*x, y\rangle$, and so $\langle (T^*T - TT^*)x, y\rangle = 0$ for all $y \in D(T)$. Since $D(T)$ is dense, we have that $(T^*T - TT^*)x = 0$ and so $T^*T = TT^*$.

[(i) $\Rightarrow$ (ii)]. Suppose $T^*T = TT^*$ and T is closed. Then $D(T^*T) = D(TT^*)$. Let $x \in D(T^*T) = D(TT^*)$. Then

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle = \langle TT^*x, x\rangle = \langle T^*x, T^*x\rangle = \|T^*x\|^2$$

and so we have that $\|Tx\| = \|T^*x\|$ whenever $x \in D(T^*T) = D(TT^*)$.

First, we would like to show $D(T^*T) = D(TT^*)$ is dense in $D(T)$ with respect to the T-norm. Let $S : D(T^*T)(\subseteq (D(T), \|\cdot\|_T)) \to \mathcal{H}$ where $S = T^*T + 1$. Further, define $J : D(T)(\subseteq \mathcal{H}) \to (D(T), \|\cdot\|_T)$ by $Jx = x$. Since $D(T)$ is dense in $\mathcal{H}$, the map J is densely defined. We claim that J is closed. Indeed, if $(x_n)_{n\geq 1} \subseteq D(T)$ such that $x_n \xrightarrow{\|\cdot\|} x \in \mathcal{H}$ and $Jx_n \xrightarrow{\|\cdot\|_T} y \in D(T)$ then, since $Jx_n = x_n$, we have that $x_n \xrightarrow{\|\cdot\|_T} y$. Then

$$\|x_n - x\|^2 \leq \|x_n - y\|^2 + \|T(x_n - y)\|^2 = \|x_n - y\|_T^2 \to 0$$

and so $x_n \xrightarrow{\|\cdot\|} y$ but $x_n \xrightarrow{\|\cdot\|} x$ and so $x = y$ and therefore $Jx = y$. Hence J is closed. Since J is densely defined and closed, J^* is densely defined. Next, we claim that $S = J^*$. Note that $D(S) = D(T^*T)$. If $y \in D(S) = D(T^*T)$ then, for any $x \in D(T)$,

$$\langle x, y\rangle + \langle Tx, Ty\rangle = \langle x, y\rangle + \langle x, T^*Ty\rangle = \langle x, (T^*T + 1)y\rangle = \langle x, Sy\rangle$$

and so $S \subseteq J^*$. Let $y \in D(J^*)$. Then there exists $z \in \mathcal{H}$ such that $\langle Jx, y\rangle_T = \langle x, z\rangle$ for all $x \in D(J) = D(T)$. Let $x \in D(T)$. Then

$$\langle x, z\rangle = \langle Jx, y\rangle_T = \langle x, y\rangle_T = \langle x, y\rangle + \langle Tx, Ty\rangle$$

and so $\langle Tx, Ty\rangle = \langle x, z-y\rangle$ and so $Ty \in D(T^*)$, hence $y \in D(T^*T)$. Therefore we have that $J^* = S$. Since J^* is dense, we then have that $D(T^*T) = D(J^*)$ is dense in $(D(T), \|\cdot\|_T)$. Let $x \in D(T)$. Then there exists $(x_n)_{n\geq 1} \subseteq D(T^*T) = D(TT^*)$ such that $x_n \xrightarrow{\|\cdot\|_T} x$. Also, since

$$\|x_n - x\|^2 \leq \|x_n - x\|^2 + \|T(x_n - x)\|^2 = \|x_n - x\|_T^2 \to 0$$

we have that $x_n \xrightarrow{\|\cdot\|} x$. Further,

$$\|Tx_n - Tx\|^2 \leq \|x_n - x\|^2 + \|T(x_n - x)\|^2 = \|x_n - x\|_T^2 \to 0$$

so $Tx_n \xrightarrow{\|\cdot\|} Tx$. Let $m, n \in \mathbb{N}$. Then

$$\|T^*x_n - T^*x_m\| = \|T^*(x_n - x_m)\| = \|T(x_n - x_m)\| = \|Tx_n - Tx_m\| \to 0.$$

Hence $(T^*x_n)_{n\geq 1}$ is $\|\cdot\|$-Cauchy and so there exists $z \in \mathcal{H}$ so that $T^*x_n \xrightarrow{\|\cdot\|} z$. Since T^* is closed, we have that $x \in D(T^*)$ and $T^*x_n \xrightarrow{\|\cdot\|} T^*x$. So $D(T) \subseteq D(T^*)$ and, since $\|Tx_n\| \to \|Tx\|$, $\|T^*x_n\| \to \|T^*x\|$, and $\|Tx_n\| = \|T^*x_n\|$ for all $n \in \mathbb{N}$, we have that $\|Tx\| = \|T^*x\|$. for all $x \in D(T) \subseteq D(T^*)$.

Now, if we instead define $V : D(TT^*)(\subseteq (D(T^*), \|\cdot\|_{T^*}) \to \mathcal{H}$ where $V = TT^* + 1$ (V should be compared to S) and $I : D(T^*)(\subseteq \mathcal{H}) \to (D(T^*), \|\cdot\|_{T^*})$ by $Ix = x$ (I should be compared to J) then similar arguments to the above will lead to the facts that $D(T^*) \subseteq D(T)$ and $\|Tx\| = \|T^*x\|$ for all $x \in D(T^*)$. Therefore, $D(T) = D(T^*)$ and $\|Tx\| = \|T^*x\|$ for all $x \in D(T) = D(T^*)$. $\square$

References

[1] Sunder, V. S.: Functional Analysis: Spectral Theory. Birkhauser, Berlin (1998)
[2] Weidmann, J.: Linear Operators in Hilbert Spaces. Springer-Verlag, New York (1980)

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