# THE VECTOR-VALUED MAXIMAL FUNCTION

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Let $f$ be a function from $\mathbb{R}^n$ into $\ell_2(\mathbb{R})$, i.e., $f = (f_1, f_2, \dots)$ where $f_i : \mathbb{R}^n \to \mathbb{R}$ for all $i = 1, 2, \dots$ and $\sum_{n=1}^{\infty} |f_n(x)|^2 < \infty$ for all $x \in \mathbb{R}^n$. Let $\mathcal{F}(X,Y)$, where $X$ and $Y$ are sets, denote the set of all functions from $X$ to $Y$. Then we define the vector-valued maximal function, $\overline{M} : \mathcal{F}(\mathbb{R}^n, \ell_2(\mathbb{R})) \to \mathcal{F}(\mathbb{R}^n, \mathbb{R})$ where

$$\overline{M}f(x) = \left( \sum_{j=1}^{\infty} (Mf_j(x))^2 \right)^{1/2} \qquad \text{for all } x \in \mathbb{R}^n$$

where $M$ is the Hardy-Littlewood Maximal function, i.e.,

$$Mf(x) = \sup_B \frac{1}{\mu(B)} \int_B |f(y)| dy$$

where the supremum is taken over all open balls containing $x$. We say a function $f : \mathbb{R}^n \to \ell_2(\mathbb{R})$ is an element of $L_p(\mathbb{R}^n, \ell_2(\mathbb{R}))$ if each $f_j$ is measurable and $\|f(x)\|_2 \in L_p(\mathbb{R}^n, \mathbb{R})$. In this case, we let $\|f\|_p = \| \|f(\cdot)\|_2 \|_p$. The purpose of this paper is to show that $\overline{M}$ is weak-(1,1) and strong-(p,p) for all $1 < p < \infty$, i.e., we want to prove the following theorem.

**Theorem 1.** (a) If $f \in L_p(\mathbb{R}^n, \ell_2(\mathbb{R}))$, $1 \le p < \infty$, then $\overline{M}f$ is finite almost everywhere.
(b) If $f \in L_1(\mathbb{R}^n, \ell_2(\mathbb{R}))$ then, for every $\alpha > 0$,

$$\mu\left(\{x : \overline{M}f(x) > \alpha\}\right) \le \frac{A}{\alpha} \|f\|_1$$

(c) If $f \in L_p(\mathbb{R}^n, \ell_2)$, $1 < p < \infty$, then $\overline{M}f \in L_p(\mathbb{R}^n, \mathbb{R})$ and $\|\overline{M}f\|_p \le A_p \|f\|_p$ for some constant $A_p$.

To prove this theorem, we first need to prove some lemmas and recall some facts about the Hardy-Littlewood Maximal function.

**Theorem 2.** Let $f$ be a function from $\mathbb{R}^n$ into $\mathbb{R}$.
(a) If $f \in L_p(\mathbb{R}^n)$, $1 \le p \le \infty$, then $Mf$ is finite almost everywhere.
(b) If $f \in L_1(\mathbb{R}^n)$, then for every $\alpha > 0$,

$$\mu\left(\{x : Mf(x) > \alpha\}\right) \le \frac{c}{\alpha} \|f\|_1$$

(c) If $f \in L_p(\mathbb{R}^n)$, $1 < p \le \infty$, then $Mf \in L_p(\mathbb{R}^n)$ and $\|Mf\|_p \le A_p \|f\|_p$ where $A_p$ is a constant depending on $c$ and $p$.

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The proof of Theorem 2 is fairly straightforward. First show $M$ is strong-($\infty$, $\infty$) and weak-(1,1) (which is the statement (b)) then use Marcenkiewicz's Theorem to show that $M$ is strong-(p,p), for $1 < p \leq \infty$, which gives us (c). Statements (b) and (c) then give (a). The proof that $M$ is strong-($\infty$, $\infty$) is trivial. The proof that $M$ is weak-(1,1) just follows from Chebyshev and a simple covering lemma which we will now state since we will need it later on. The proof of the lemma involves a basic greedy argument.

**Lemma 3.** Let $E$ be a measurable subset of $\mathbb{R}^n$ that is the union of a finite collection of balls $\{B_k\}_{k=1}^n = \{B_k(x_k, \epsilon_k)\}_{k=1}^n$. Then there exists a disjoint subcollection $\{B_{k_j}\}_{j=1}^m$ of $\{B_k\}_{k=1}^n$ so that

$$\mu(E) \leq 3^n \sum_{j=1}^{m} \mu(B_{k_j})$$

We will need another covering lemma as well for the proof of Theorem 1, so let us state it now and give its proof.

**Lemma 4: Calderon-Zygmund Lemma.** Let $f : \mathbb{R}^n \to \mathbb{C}$ be non-negative and integrable and let $\alpha > 0$. Then there exists sets $F$ and $\Omega$ such that:
(1) $\mathbb{R}^n = F \cup \Omega$,
(2) $|f(x)| \leq \alpha$ almost everywhere on $F$, and
(3) $\Omega$ is a union of cubes; $\Omega = \cup_k Q_k$ whose interiors are mutually disjoint, and so that, for each $Q_k$, we have that

$$\alpha < \frac{1}{\mu(Q_k)} \int_{Q_k} f(x)dx \leq 2^n \alpha$$

*Proof.* Since $f \in L_1(\mathbb{R}^n)$, we have that $\int_{R^n} f(x)dx = c < \infty$ so choose $k \in \mathbb{Z}$ so that, if $Q$ is a cube in $R^n$ with edge lengths $2^k$ then $\mu(Q) > \frac{c}{\alpha}$. Then,

$$\frac{1}{\mu(Q)} \int_{Q} f(x)dx \leq \frac{1}{\mu(Q)} \int_{\mathbb{R}^n} f(x)dx \leq \frac{\alpha}{c} c = \alpha$$

Now, divide $\mathbb{R}^n$ into cubes with disjoint interiors whose side lengths are all $2^k$ and let $Q$ be such a cube. Divide $Q$ into $2^n$ congruent cubes by bisecting each of its sides. Let $Q'$ be one of our new subcubes. If $\frac{1}{\mu(Q')} \int_{Q'} f(x)dx > \alpha$ then let $Q'$ be one of the $Q_k$ mentioned in the statement of the proof. Note that

$$\alpha < \frac{1}{\mu(Q')} \int_{Q'} f(x)dx \leq \frac{1}{\mu(Q')} \int_{Q} f(x)dx = 2^n \frac{1}{\mu(Q)} \int_{Q} f(x)dx \leq 2^n \alpha$$

If instead, we have that $\frac{1}{\mu(Q')} \int_{Q'} f(x)dx \leq \alpha$ then divide $Q'$ into $2^n$ congruent cubes and repeat the process. Let $\Omega = \cup_k Q_k$ and $F = \Omega^c$. Let $x \in F$. Then, for any cube $Q$ constructed in our process which contains $x$, we have that $\frac{1}{\mu(Q)} \int_{Q} f(x)dx \leq \alpha$ and so

$$f(x) = \lim_{diam(Q) \to 0} \frac{1}{\mu(Q)} \int_{Q} f(y)dy \leq \alpha \quad \text{a.e.}$$

Where the $Q$'s in the limit can be taken as a sequence of cubes from our construction containing x whose diameters are converging to zero. This completes the proof of our lemma.

□

We have two more Lemmas to consider before proceeding to the proof of Theorem 1.

**Lemma 5.** Let $Q_1$ and $Q_2$ be cubes in $\mathbb{R}^n$ such that $Q_1 \cap Q_2 \neq \phi$. Let $\widetilde{Q}$ denote the cube with the same center as $Q$ but with $2n$-times the diameter. Then, either $Q_1 \subseteq \widetilde{Q_2}$ or $Q_2 \subseteq \widetilde{Q_1}$.

*Proof.* Suppose $diam(Q_1) \leq diam(Q_2)$, then we claim that $Q_1 \subseteq \widetilde{Q_2}$. The case when $n = 1$ is trivial so suppose $n \geq 2$. Also, we can assume that $Q_2$ is centered at the origin. Let $Q_2 = [\frac{-s}{2}, \frac{s}{2}]^n$ where $s$ is the length of the edges of $Q_2$. Then $\widetilde{Q_2} = [-sn, sn]^n$. Since $Q_1 \cap Q_2 \neq \phi$, let $y \in Q_1 \cap Q_2$. Let $y = (y_1, y_2, \ldots, y_n)$. Since $y \in Q_2$ we have that $\frac{-s}{2} \leq y_i \leq \frac{s}{2}$ for all $i = 1, \ldots, n$. Let $x = (x_1, x_2, \ldots, x_n) \in Q_1$. Since $y \in Q_1$ we have that $d(x, y) \leq diam(Q_1) \leq diam(Q_2)$. Suppose $x \notin \widetilde{Q_2}$. Then there exists $k = 1, \ldots, n$ so that $|x_k| > sn$. Since $|x_k| > sn$ and $|y_k| \leq \frac{s}{2}$ we have that

$$|x_k - y_k| \geq ||x_k| - |y_k|| \geq |sn - \frac{s}{2}|$$

and so $(x_k - y_k)^2 \geq (sn - \frac{s}{2})^2$. Let $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots b_n) \in Q_2$. Then,

$$\begin{aligned} d(a,b)^2 &= (a_1 - b_1)^2 + (a_2 - b_2)^2 + \cdots + (a_n - b_n)^2 \\ &\leq (\frac{s}{2} + \frac{s}{2})^2 + (\frac{s}{2} + \frac{s}{2})^2 + \cdots + (\frac{s}{2} + \frac{s}{2})^2 \\ &= s^2 n \end{aligned}$$

So, $diam(Q_2) \leq s\sqrt{n}$. By considering $a = (\frac{s}{2}, \frac{s}{2}, \ldots, \frac{s}{2})$ and $b = (\frac{-s}{2}, \frac{-s}{2}, \ldots, \frac{-s}{2})$ we see that we actually have $diam(Q_2) = s\sqrt{n}$. Then,

$$d(x,y) = \big((x_1 - y_1)^2 + \cdots + (x_n - y_n)^2\big)^{1/2} \geq \big((x_k - y_k)^2\big)^{1/2} \geq sn - \frac{s}{n} = s(n - \frac{1}{2}) > s\sqrt{n} = diam(Q_2)$$

where $s(n - \frac{1}{2}) > s\sqrt{n}$ because $n > 2$. This clearly contradicts the fact that $d(x, y) \leq diam(Q_2)$ so we must have that $x \in \widetilde{Q_2}$ and the proof is complete.

□

**Lemma 6.** Let $f$ and $\phi$ be positive real-valued functions on $\mathbb{R}^n$ and let $q > 1$. Then

$$\int_{\mathbb{R}^n} (Mf(x))^q \phi(x)dx \leq B_q \int_{\mathbb{R}^n} |f(x)|^q M\phi(x)dx$$

for some constant $B_q$. Further,

$$\int_{\{x: Mf(x) > \alpha\}} \phi(x)dx \leq \frac{A}{\alpha} \int_{\mathbb{R}^n} |f(x) M\phi(x)dx$$

for some constant A.

*Proof.* Let $\phi$ be a positive real-valued function on $\mathbb{R}^n$. We can then define a measure, say $\nu$, on $\mathbb{R}^n$ by letting $v(E) = \int_E \phi(x)dx$ for all $\mu$-measurable sets $E$. Note then, for a function $f$, that $\int_{\mathbb{R}^n} f(x)d\nu(x) = \int_{\mathbb{R}^n} f(x)\phi(x)dx$. Similarly, we can define a measure $\rho$ on $\mathbb{R}^n$ by letting $\rho(E) = \int_E M\phi(x)dx$. Consider

$M : L_\infty(\mathbb{R}^n, \rho) \to L_\infty(\mathbb{R}^n, \nu)$. Let $f \in L_\infty(\mathbb{R}^n, \rho)$. Then

$$\begin{aligned}\|Mf\|_{L_\infty(\nu)} &= \text{ess} \sup_{x\in\mathbb{R}^n} \sup_{x\in Q} \frac{1}{\mu(Q)} \int_Q |f(y)|dy \\ &\leq \text{ess} \sup_{x\in\mathbb{R}^n} \sup_{x\in Q} \frac{1}{\mu(Q)} \int_Q \|f\|_{L_\infty(\rho)} dy \\ &= \text{ess} \sup_{x\in\mathbb{R}^n} \sup_{x\in Q} \frac{1}{\mu(Q)} \|f\|_{L_\infty(\rho)} \mu(Q) \\ &= \|f\|_{L_\infty(\rho)}\end{aligned}$$

So we have that $M$ is strong-($\infty$, $\infty$) with respect to $\rho$ and $\nu$. Let $E_\alpha = \{x : Mf(x) > \alpha\}$ and let $E$ be a compact subset of $E_\alpha$. Let $x \in E$. Then there exists $B(x, \epsilon_x)$ so that $x \in B(x, \epsilon_x)$ and

$$\frac{1}{\mu(B(x,\epsilon_x))} \int_{B(x,\epsilon_x)} |f(y)|dy > \alpha$$

which gives that

$$\mu(B(x, \epsilon_x)) < \frac{1}{\alpha} \int_{B(x,\epsilon_x)} |f(y)|dy$$

Then, $\{B(x, \epsilon_x\}$ is an open cover of $E$, so there exists a finite subecover, say $B(x_1, \epsilon_1), \ldots, B(x_k, \epsilon_k)$. Then, by Lemma 3, we can select a disjoint collection $B(x_1, \epsilon_1), \ldots, B(x_m, \epsilon_m)$ (where we allow for re-indexing) of the above finite collection, so that $E \subseteq \cup_{j=1}^m B(x_j, 3\epsilon_j)$. Well,

$$\int_B |f(y)|M\phi(y)dy \geq \int_B |f(y)| \left(\frac{1}{\mu(B)} \int_B \phi(z)dz\right) dy = \left(\frac{1}{\mu(B)} \int_B |f(y)|dy\right) \left(\int_B \phi(z)dz\right) \geq \alpha \int_B \phi(z)dz$$

Then, we have

$$\begin{aligned}\nu(E) &\leq \sum_{j=1}^m \nu\left(B(x_j, 3\epsilon_j)\right) \\ &\leq A \sum_{j=1}^m \nu\left(B(x_j, \epsilon_j)\right) \\ &= A \sum_{j=1}^m \int_{B(x_j,\epsilon_j)} \phi(y)dy \\ &\leq A \sum_{j+1}^m \frac{1}{\alpha} \int_{B(x_j,\epsilon_j)} |f(y)|M\phi(y)dy \qquad \text{by above} \\ &\leq \frac{A}{\alpha} \int_{\mathbb{R}^n} |f(y)|M\phi(y)dy \\ &= \frac{A}{\alpha} \|f\|_{L_1(\rho)}\end{aligned}$$

Since this is true for all compact subsets of $E_\alpha$, we have that $\nu(E_\alpha) \leq \frac{A}{\alpha}\|f\|_{L_1(\rho)}$, and so $M$ is weak-(1,1) with respect to $\nu$ and $\mu$. Therefore, by Marcinkiewicz, we have that $M : L_p(\mathbb{R}^n, \rho) \to L_p(\mathbb{R}^n, \nu)$ is strong-(p,p) for all $1 < p \leq \infty$, i.e.,

$$\left(\int_{\mathbb{R}^n} (Mf(y))^p\, d\nu(y)\right)^{1/p} \leq c \left(\int_{\mathbb{R}^n} |f(y)|^p d\rho(y)\right)^{1/p}$$

And so,

$$\int_{\mathbb{R}^n} (Mf(x))^p \phi(x)dx \leq c_p \int_{\mathbb{R}^n} |f(x)|^p M\phi(x)dx$$

which completes the proof of Lemma 6.

□

Before proceeding to the proof of Theorem 1, a remark is in order. For a Banach space $X$, let $X^*$ denote the dual of $X$, i.e., $X^*$ is the set of all bounded linear functionals on $X$. Let $p, q \geq 1$ be Holder conjugates. Then it is a well known fact that $L_p(\mathbb{R}^n)$ is isometrically isomorphic to $L_q(\mathbb{R}^n)^*$. Further, for $f \in L_p(\mathbb{R}^n)$, its "representation", say $\varphi_f$, as a member of $L_q(\mathbb{R}^n)^*$ is given by $\varphi_f(g) = \int_{\mathbb{R}^n} g(x)f(x)dx$. Let $\|\cdot\|_L$ denote the operator norm on $L_q(\mathbb{R}^n)^*$. Then we have that

$$\|f\|_p = \|\varphi_f\|_L = \sup\{|\varphi_f(g)| : g \in L_q(\mathbb{R}^n),\ \|g\| \leq 1\} = \sup\{|\int_{\mathbb{R}^n} g(x)f(x)dx| : g \in L_q(\mathbb{R}^n),\ \|g\| \leq 1\}$$

We will use this equality later in the proof of Theorem 1. Let us restate the theorem again.

**Theorem 1.** (a) If $f \in L_p(\mathbb{R}^n, \ell_2(\mathbb{R}))$, $1 \leq p < \infty$, then $\overline{M}f$ is finite almost everywhere.
(b) If $f \in L_1(\mathbb{R}^n, \ell_2(\mathbb{R}))$ then, for every $\alpha > 0$,

$$\mu\left(\{x : \overline{M}f(x) > \alpha\}\right) \leq \frac{A}{\alpha}\|f\|_1$$

(c) If $f \in L_p(\mathbb{R}^n, \ell_2)$, $1 < p < \infty$, then $\overline{M}f \in L_p(\mathbb{R}^n, \mathbb{R})$ and $\|\overline{M}f\|_p \leq A_p\|f\|_p$ for some constant $A_p$.

*Proof.* We will start by proving (c) for $p = 2$.

$$\begin{aligned}
\|\overline{M}f\|_2^2 &= \int_{\mathbb{R}^n} \left(\left(\sum_{j=1}^{\infty} (Mf_j(x))^2\right)^{1/2}\right)^2 d\mu \\
&= \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} (Mf_j(x))^2\, d\mu && \text{MCT} \\
&= \sum_{j=1}^{\infty} \|Mf_j\|_2^2 \\
&\leq A \sum_{j=1}^{\infty} \|f_j\|_2^2 && \text{Theorem 2} \\
&= A \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} |f_j(x)|^2 d\mu \\
&= A \int_{\mathbb{R}^n} \sum_{j=1}^{\infty} |f_j(x)|^2 d\mu && \text{MCT} \\
&= \int_{\mathbb{R}^n} \|f(x)\|_2^2 d\mu \\
&= A\|f\|_2^2
\end{aligned}$$

We now want to prove (b). It suffices to show (b) when $f_j(x) \geq 0$ for all $x$. Apply the Calderon-Zygmund lemma to the function $\|f(\cdot)\|_2$. Then we have disjoint sets $F$ and $\Omega$ so that $\mathbb{R}^n = F \cup \Omega$, $\|f(x)\|_2 \leq \alpha$ almost everywhere on $F$, and $\Omega = \cup_k Q_k$ where the $Q_k$'s are cubes whose interiors are mutually disjoint and so that, for each $Q_k$ we have

$$\alpha < \frac{1}{\mu(Q_k)} \int_{Q_k} \|f(x)\|_2 dx \leq 2^n \alpha \tag{1}$$

Note that the above gives

$$\mu(Q_k) < \frac{1}{\alpha} \int_{Q_k} \|f(x)\|_2 dx$$

And so

$$\mu(\Omega) = \sum_k \mu(Q_k) \leq \frac{1}{\alpha} \sum_k \int_{Q_k} \|f(x)\|_2 dx \leq \frac{1}{\alpha} \int_{\mathbb{R}^n} \|f(x)\|_2 dx = \frac{1}{\alpha} \|f\|_1 \tag{2}$$

Let $f_k' = f_k 1_F$ and $f_k'' = f_k 1_\Omega$. Let $f' = (f_1', f_2', \dots)$ and $f'' = (f_1'', f_2'', \dots)$.
Claim 1:

$$\mu\left(\{x \in \mathbb{R}^n : \overline{M} f'(x) > \alpha\}\right) \leq \frac{A}{\alpha} \|f\|_1$$

Clearly $\|f'(x)\|_2 \leq \|f(x)\|_2$ so

$$\|f'\|_2^2 = \int_{\mathbb{R}^n} \|f'(x)\|_2^2 dx \leq \int_F \|f(x)\|_2^2 dx \leq \alpha \int_F \|f(x)\|_2 dx \leq \alpha \|f\|_1$$

where we used the fact that $\|f(x)\|_2 \leq \alpha$ on $F$. Then we have that

$$\|\overline{M} f'\|_2^2 \leq A \|f'\|_2^2 \leq \alpha \|f\|_1 \tag{3}$$

Note that for a square-integrable function $g$,

$$\|g\|_2^2 = \int_{\mathbb{R}^n} |g(x)| dx \geq \int_{x:|f(x)|>\alpha} |f(x)|^2 dx \geq \int_{x:|f(x)|>\alpha} \alpha^2 dx = \alpha^2 \mu\left(\{x : |f(x)| > \alpha\}\right) \tag{4}$$

So, if we apply this to $\overline{M}(f')$ and use (3), we get

$$\mu\left(\{x \in \mathbb{R}^n : \overline{M} f'(x) > \alpha\}\right) \leq \frac{1}{\alpha^2} \|\overline{M} f'\|_2^2 \leq \frac{1}{\alpha} \|f\|_1$$

Claim 2:

$$\mu\left(\{x \in \mathbb{R}^n : \overline{M} f''(x) > \alpha\}\right) \leq \frac{A}{\alpha} \|f\|_1$$

Let

$$\overline{f}_j(x) = \begin{cases} \frac{1}{\mu(Q_j)} \int_{Q_j} |f_j(y)| dy & \text{if } x \in Q_j \\ 0 & \text{if } x \in F \end{cases}$$

and let $\overline{f} = (\overline{f}_1, \overline{f}_2, \dots)$. Let $x \in Q_j$. Then

$$\|\overline{f}(x)\|_2 = \left( \sum_{k=1}^{\infty} \left( \frac{1}{\mu(Q_j)} \int_{Q_j} |f_k(y)| dy \right)^2 \right)^{1/2} = \frac{1}{\mu(Q_j)} \| \int_{Q_j} |f_\cdot(y)| dy \|_2 \leq \frac{1}{\mu(Q_j)} \int_{Q_j} \|f_\cdot(y)\|_2 dy$$

where the last inequality comes from Generalized Minkowski. Further, by using (1) we get

$$\frac{1}{\mu(Q_j)} \int_{Q_j} \|f_\cdot(y)\|_2 dy = \frac{1}{\mu(Q_j)} \int_{Q_j} \left( \sum_{k=1}^{\infty} |f_k(y)|^2 \right)^{1/2} dy \leq c\alpha$$

and so $\|\overline{f}(x)\|_2 \leq c\alpha$. If $x \notin \Omega$ then $\overline{f}_k(x) = 0$ for all $k \in \mathbb{N}$ and so $\|\overline{f}(x)\|_2 = 0$. Then

$$\|\overline{f}\|_2^2 = \int_{\mathbb{R}^n} \|\overline{f}(x)\|_2^2 dx = \int_\Omega \|\overline{f}(x)\|_2^2 dx \leq \int_\Omega c^2 \alpha^2 dx = c^2 \alpha^2 \mu(\Omega) \leq c^2 \alpha \|f\|_1$$

where we used (2) for the last inequality. We know from the beginning of the proof that $\|\overline{Mf}\|_2^2 \leq A\|\overline{f}\|_2^2$ so we have that $\|\overline{Mf}\|_2^2 \leq A\|\overline{f}\|_2^2 \leq Ac^2\alpha\|\overline{f}\|_1$. Then by (4), we have

$$\mu\left(\{x \in \mathbb{R}^n : \overline{Mf}(x) > \alpha\}\right) \leq \frac{1}{\alpha^2}\|\overline{Mf}\|_2^2 \leq \frac{Ac^2}{\alpha}\|f\|_1 \tag{5}$$

Let $Q$ be a cube in $\mathbb{R}^n$. Let $\widetilde{Q}$ denote a cube with the same center as $Q$ but with $2n$-times the diameter. Let $\widetilde{\Omega} = \cup_j \widetilde{Q_j}$. Then, using (2), we have

$$\mu(\widetilde{\Omega}) = \mu(\cup_j \widetilde{Q_j}) \leq \sum_j \mu(\widetilde{Q_j}) \leq c\sum_j \mu(Q_j) \leq \frac{c}{\alpha}\|f\|_1 \tag{6}$$

Claim: If $x \notin \widetilde{\Omega}$ then $Mf_k''(x) \leq cM\overline{f}_k(x)$.

Let $Q$ be a cube containing $x$. Let $J = \{j : Q_j \cap Q \neq \phi\}$. Since $\cup_j Q_j = \Omega$, where the $Q_j$'s have disjoint interior, we have that $Q \cap \Omega = \cup)j \in J(Q \cap Q_j)$ and $\mu(Q \cap \Omega) = \sum_{j\in J}\mu(Q \cap Q_j)$. So

$$\begin{aligned}
\frac{1}{\mu(Q)}\int_Q |f_k''(y)|dy &= \frac{1}{\mu(Q)}\int_{Q\cap\Omega} |f_k''(y)|dy \\
&= \frac{1}{\mu(Q)}\sum_{j\in J}\int_{Q_j\cap Q} |f_k''(y)|dy \\
&\leq \frac{1}{\mu(Q)}\sum_{j\in J}\int_{Q_j} |f_k''(y)|dy \\
&= \frac{1}{\mu(Q)}\sum_{j\in J}\int_{Q_j} |f_k(y)|dy \\
&= \frac{1}{\mu(Q)}\sum_{j\in J}\frac{1}{\mu(Q_j)}\int_{Q_j} |f_k(y)|dy\int_{Q_j} 1dz \\
&= \frac{1}{\mu(Q)}\sum_{j\in J}\int_{Q_j}\frac{1}{\mu(Q_j)}\int_{Q_j} |f_k(y)|dydz \\
&= \frac{1}{\mu(Q)}\sum_{j\in J}\int_{Q_j}\overline{f}_k(z)dz
\end{aligned}$$

And, for $j \in J$, we have that $Q_j \cap Q \neq \phi$ and $x \in Q - \widetilde{\Omega} \subseteq Q - \widetilde{Q_j}$ so by Lemma 5 we have that $Q_j \subseteq \widetilde{Q}$. Then,

$$\frac{1}{\mu(Q)}\sum_{j\in J}\int_{Q_j}\overline{f}_k(z)dz \leq \frac{1}{\mu(Q)}\int_{\widetilde{Q}}\overline{f}_k(z)dz \leq \frac{c}{\mu(\widetilde{Q})}\int_{\widetilde{Q}}\overline{f}_k(z)dz \leq cM\overline{f}_k(x)$$

So we have that, for all $Q$ containing $x$, $\frac{1}{\mu(Q)}\int_Q |f_k''(y)|dy \leq cM\overline{f}_k(x)$ and so

$$Mf_k''(x) \leq cM\overline{f}_k(x) \tag{7}$$

for all $x \in \mathbb{R}^n - \widetilde{\Omega}$. Then, by (5), (6), and (7), we have

$$\begin{aligned}\mu\left(\{x \in \mathbb{R}^n : \overline{M}f''(x) > c\alpha\}\right) &\leq \mu\left(\{x \in \mathbb{R}^n - \Omega : \overline{M}f''(x) > c\alpha\}\right) + \mu(\Omega) \\ &\leq \mu\left(\{x \in \mathbb{R}^n - \Omega : \overline{Mf}(x) > \alpha\}\right) + \mu(\Omega) \\ &\leq \frac{A}{\alpha}\|f\|_1 + \frac{c'}{\alpha}\|f\|_1 \\ &= \frac{c''}{\alpha}\|f\|_1\end{aligned}$$

which finishes the proof of claim 2. Now we are ready to prove (b). Recall that $f' = f1_f$ and $f'' = f1_\Omega$ where $F \cup \Omega = \mathbb{R}^n$ and $F \cap \Omega = \phi$. So, for $x \in \mathbb{R}^n$, either $f(x) = f'(x)$ or $f(x) = f''(x)$. So

$$\overline{M}f(x) = \begin{cases} \overline{M}f'(x) & \text{if } x \in F \\ \overline{M}f''(x) & \text{if } x \in \Omega \end{cases}$$

So,

$$\{x : \overline{M}f(x) > \alpha\} \subseteq \{x : \overline{M}f'(x) > \alpha\} \cup \{x : \overline{M}f''(x) > \alpha\}$$

And so,

$$\mu\left(\{x : \overline{M}f(x) > \alpha\}\right) \leq \mu\left(\{x : \overline{M}f'(x) > \alpha\}\right) + \mu\left(\{x : \overline{M}f''(x) > \alpha\}\right) \leq \frac{c}{\alpha}\|f\|_1$$

which completes the proof of (b). We have now shown that $\overline{M}$ is weak-(1,1) and strong-(2,2) so, by Marcinkiewicz, we have that $\overline{M}$ is strong-(p,p) for $1 < pleq2$ so we are left with proving (c) for the case when $p \geq 2$. By Lemma 6, for $q = 2$, we have that, for all $j$,

$$\int_{\mathbb{R}^n} (Mf_j(x))^2 \phi(x) dx \leq A \int_{\mathbb{R}^n} |f_j(x)|^2 M\phi(x) dx$$

And so

$$\sum_{j=1}^{\infty} \int_{\mathbb{R}^n} (Mf_j(x))^2 \phi(x) dx \leq A \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} |f_j(x)|^2 M\phi(x) dx$$

Further,

$$\sum_{j=1}^{\infty} \int_{\mathbb{R}^n} (Mf_j(x))^2 \phi(x) dx = \int_{\mathbb{R}^n} \sum_{j=1}^{\infty} (Mf_j(x))^2 \phi(x) dx = \int_{\mathbb{R}^n} \left(\overline{M}f(x)\right)^2 \phi(x) dx$$

by the Monotone Convergence Theorem. Also,

$$A \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} |f_j(x)|^2 M\phi(x) dx = A \int_{\mathbb{R}^n} \sum_{j=1}^{\infty} |f_j(x)|^2 M\phi(x) dx = A \int_{\mathbb{R}^n} \|f(x)\|_2^2 M\phi(x) dx$$

again, by the Monotone Convergence Theorem, and so we have that

$$\int_{\mathbb{R}^n} \left(\overline{M}f(x)\right)^2 \phi(x) dx \leq A \int_{\mathbb{R}^n} \|f(x)\|_2^2 M\phi(x) dx$$

Let $r$ be the Holder conjugate of $\frac{p}{2}$ (Note: $\frac{p}{2} \geq 1$) since $p \geq 2$. Then the above gives

$$\sup_{\phi \in L_r : \|\phi\| \leq 1} \int_{\mathbb{R}^n} \left(\overline{M}f(x)\right)^2 \phi(x) dx \leq \sup_{\phi \in L_r : \|\phi\| \leq 1} A \int_{\mathbb{R}^n} \|f(x)\|_2^2 M\phi(x) dx \tag{8}$$

From our discussion about the dual space of $L_{p/2}$ prior to this theorem, we know

$$\sup_{\phi \in L_r : \|\phi\| \leq 1} \int_{\mathbb{R}^n} \left(\overline{M}f(x)\right)^2 \phi(x) dx = \|\left(\overline{M}f\right)^2\|_{p/2} = \|\overline{M}f\|_p^2$$

So from (8) we have

$$\begin{aligned}
\|\overline{M}f\|_p^2 &\leq \sup_{\phi \in L_r : \|\phi\| \leq 1} A \int_{\mathbb{R}^n} \|f(x)\|_2^2 M\phi(x) dx \\
&\leq \sup_{\phi \in L_r : \|\phi\| \leq 1} A \left( \int_{\mathbb{R}^n} \|f(x)\|_2^{2\frac{p}{2}} dx \right)^{2/p} \left( \int_{\mathbb{R}^n} (M\phi(x))^r \, dx \right)^{1/r} \\
&= \sup_{\phi \in L_r : \|\phi\| \leq 1} A \|f\|_p^2 \|M\phi\|_r \\
&\leq \sup_{\phi \in L_r : \|\phi\| \leq 1} A \|f\|_p^2 c \|\phi\|_r \\
&\leq cA\|f\|_p^2
\end{aligned}$$

So we have that $\|\overline{M}f\|_p \leq c'\|f\|_p$, which completes the proof of Theorem 1.

□

## References

1. C. Fefferman and E.M. Stein, *Some Maximal Inequalities*, American Journal of Mathematics, Vol. 93 (1971), pp. 107-115

2. Stein, E.M. (1993), *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals*. Princeton: Princeton University Press.