THE REGULARITY THEOREM FOR DISTRIBUTIONS

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The purpose of this paper is to give a proof of the one-dimensional Regularity Theorem for Distributions which states that if T is a tempered distribution on $\mathbb{R}$ then T is the weak nth derivative of some polynomially bounded continuous function. We will start by giving the necessary definitions then prove the N-Representation Theorem for the Schwartz class and for tempered distributions which we will then use to prove the Regularity Theorem.

Definition 1. A **seminorm** on a vector space V is a map $\rho : V \to [0, \infty)$ such that

$$(i) \quad \rho(x + y) \leq \rho(x) + \rho(y)$$

$$(ii) \quad \rho(\alpha x) = |\alpha|\rho(x) \quad \text{for all } \alpha \in \mathbb{C}$$

A family of seminorms $(\rho_\alpha)_{\alpha \in A}$ is said to **separate points** if $\rho_\alpha(x) = 0$ for all $\alpha \in A$ implies $x = 0$.

Definition 2. A **locally convex space** is a vector space X with a family of seminorms $(\rho_\alpha)_{\alpha \in A}$ which separates points. The **natural topology** on a locally convex space is the weakest (or smallest) topology for which all the ρ_α are continuous and for which the operation of addition is continuous.

Definition 3. If $(\rho_\alpha)_{\alpha \in A}$ and $(d_\beta)_{\beta \in B}$ are two families of seminorms on a vector space X such that the natural topologies with respect to each family are the same then we say the families $(\rho_\alpha)_{\alpha \in A}$ and $(d_\beta)_{\beta \in B}$ on X are **equivalent**.

Proposition 1. Let $(\rho_\alpha)_{\alpha \in A}$ and $(d_\beta)_{\beta \in B}$ be two families of seminorms. Then the families are equivalent if and only if, for each $\alpha \in A$, there exists $\beta_1, \beta_2, \ldots, \beta_n \in B$ and $C > 0$ so that for all $x \in X$

$$\rho_\alpha(x) \leq C \left(d_{\beta_1}(x) + \cdots + d_{\beta_n}(x)\right)$$

and for each $\beta \in B$ there exists $\alpha_1, \alpha_2, \ldots, \alpha_m \in A$ and $D > 0$ so that for all $x \in X$

$$d_\beta(x) \leq D \left(\rho_{\alpha_1}(x) + \cdots + \rho_{\alpha_n}(x)\right)$$

Proof. First, suppose the families are equivalent. Let $\alpha \in A$. Then $\{x : \rho_\alpha(x) < 1\}$ is τ_d-open. So there exists $N = N^d_{\beta_1,\ldots,\beta_n,\epsilon}$ such that $N \subseteq \{x : \rho_\alpha(x) < 1\}$. Suppose $x \in X$ such that $d_{\beta_k} \neq 0$ for some $k = 1, \ldots n$. Then for all $k = 1, \ldots n$,

$$d_{\beta_k}\left(\frac{\epsilon x}{d_{\beta_1}(x) + \cdots + d_{\beta_n}(x)}\right) < \epsilon$$

So,

$$\rho_\alpha\left(\frac{\epsilon x}{d_{\beta_1}(x) + \cdots + d_{\beta_n}(x)}\right) < 1$$

Date: May 2013.

hence,

$$\rho_\alpha(x) < \frac{1}{\epsilon}\left(d_{\beta_1}(x) + \cdots + d_{\beta_n}(x)\right)$$

If $d_{\beta_k}(x) = 0$ for all $k = 1, \ldots, n$ then $d_{\beta_k}(ax) = 0 < \epsilon$ for all $k = 1, \ldots, n$ and so $\rho_\alpha(ax) < 1$. Hence, $\rho_\alpha(x) < \frac{1}{a}$ for all $a > 0$ and therefore $\rho_\alpha(x) = 0$. The second statement is symmetric to the first. Now, for the other direction. Let $(x_\delta)_{\delta \in D}$ be a net in X such that $x_\delta \to x$ in the τ_d -topology, i.e., $d_\beta(x_\delta - x) \to 0$ for all $\beta \in B$. Then,

$$\rho_\alpha(x_\delta - x) \leq C\left(d_{\beta_1}(x) + \cdots + d_{\beta_n}(x)\right) \to 0$$

So ρ_α is continuous with respect to the τ_d -topology for all $\alpha \in A$, hence, $\tau_\rho \subseteq \tau_d$. A symmetric argument shows $\tau_d \subseteq \tau_\rho$.

□

Definition 4. A family $(\rho_\alpha)_{\alpha \in A}$ of seminorms on a vector space V is called **directed** if and only if for all $\alpha, \beta \in A$ there is a $\gamma \in A$ and a $C > 0$ so that $\rho_\alpha(x) + \rho_\beta(x) \leq C\rho_\gamma(x)$ for all $x \in V$.

Definition 5. If X is a locally convex space then the **topological dual**, denoted by X^*, is the set of continuous linear functionals on X with respect to the natural topology.

Definition 6. The **Schwartz class**, denoted by $\mathcal{S}(\mathbb{R})$, is the set of infinitely differentiable complex-valued functions φ on $\mathbb{R}$ for which

$$\|\varphi\|_{n,m,\infty} := \sup_{x \in \mathbb{R}^n} |x^n D^m \varphi(x)| < \infty \quad \text{for all } n, m \in I_+$$

where $I_+ = \mathbb{N} \cup \{0\}$. It is easy to see that $(\|\cdot\|_{n,m,\infty})_{n,m \in I_+}$ is a family of seminorms which separates points.

Definition 7. The Space of Tempered Distributions, denoted by $\mathcal{S}'(\mathbb{R})$, is the topological dual of $\mathcal{S}(\mathbb{R})$.

Note. $\mathcal{S}(\mathbb{R})$ embeds $\sigma(\mathcal{S}', \mathcal{S})$-continuously into $\mathcal{S}'(\mathbb{R})$ where the $\sigma(\mathcal{S}', \mathcal{S})$-topology is the smallest topology on $\mathcal{S}'(\mathbb{R})$ such that the maps $\{\gamma_x : \mathcal{S}'(\mathbb{R}) \to \mathbb{C} | x \in X\}$ are continuous, where $\gamma_x(\ell) = \ell(x)$ for all $\ell \in \mathcal{S}'(\mathbb{R})$. Further, $\mathcal{S}(\mathbb{R})$ is dense in $\mathcal{S}'(\mathbb{R})$.

Definition 8. Let $T \in \mathcal{S}'(\mathbb{R})$ and $n \in I_+$. The **weak nth derivative of T**, denoted $D^n T$, is defined by

$$\left(D^n T\right)(f) = (-1)^n T\left(D^n f\right)$$

We are now ready to prove the N-Representation Theorem for $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ after a couple more definitions and lemmas.

Lemma 1. For $n, m \in I_+$ define a seminorm $\|\cdot\|_{n,m,2}$ on $\mathcal{S}(\mathbb{R})$ by

$$\|f\|_{n,m,2} = \left(\int_{\mathbb{R}} |x^n D^m f(x)|^2 dx\right)^{1/2}$$

Then the families of seminorms $(\|\cdot\|_{n,m,\infty})_{n,m \in I_+}$ and $(\|\cdot\|_{n,m,2})_{n,m \in I_+}$ on $\mathcal{S}(\mathbb{R})$ are equivalent.

Proof. Let $f \in \mathcal{S}(\mathbb{R})$ and let $g(x) = (1+x^2)^{-1}$. Then $g \in L_2(\mathbb{R})$ and, for $n, m \in I_+$,

$$\begin{aligned}
\|f\|_{n,m,2} &= \|x^n D^m f(x)\|_2 \\
&= \|\frac{(1+x^2)x^n D^m f(x)}{1+x^2}\|_2 \\
&= \left(\int_{\mathbb{R}} \left|\frac{(1+x^2)x^n D^m f(x)}{1+x^2}\right|^2 dx\right)^{1/2} \\
&\leq \left(\int_{\mathbb{R}} \frac{\|(1+x^2)x^n D^m f(x)\|_\infty^2}{|1+x^2|^2} dx\right)^{1/2} \\
&= \|(1+x^2)x^n D^m f(x)\|_\infty \|g\|_2 \\
&\leq \|g\|_2 \left(\|x^n D^m f(x)\|_\infty + \|x^{n+2} D^m f(x)\|_\infty\right) \\
&\leq C\left(\|f\|_{n,m,\infty} + \|f\|_{n+2,m,\infty}\right)
\end{aligned}$$

Further, for any $f \in \mathcal{S}(\mathbb{R})$,

$$\|f\|_\infty = \sup_{x\in\mathbb{R}} \left|\int_{-\infty}^{x} f'(x)dx\right| \leq \int_{\mathbb{R}} |f'(x)|dx = \|f'\|_1 \leq \|(1+x^2)f'\|_2 \|(1+x^2)^{-1}\|_2$$

So we have that

$$\begin{aligned}
\|f\|_{n,m,\infty} &\leq \|(1+x^2)\frac{d}{dx}(x^n D^m f(x))\|_2 \|(1+x^2)^{-1}\|_2 \\
&= C\|(1+x^2)(nx^{n-1}D^m f(x) + x^n D^{m+1} f(x)\|_2 \\
&\leq C\left(\|nx^{n-1}D^m f(x)\|_2 + \|x^n D^{m+1} f(x)\|_2 + \|nx^{n+1}D^m f(x)\|_2 + \|x^{n+2}D^{m+1}f(x)\|_2\right) \\
&= C'\left(\|f\|_{n-1,m,2} + \|f\|_{n,m+1,2} + \|f\|_{n+1,m,2} + \|f\|_{n+2,m+1,2}\right)
\end{aligned}$$

So, by Proposition 1, the families of seminorms are equivalent. □

Definition 9. Let $A : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ and $A^* : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ where

$$A = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right) \quad \text{and} \quad A^* = \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right)$$

Let $N = A^*A$ and define a seminorm $\|\cdot\|_n^*$ on $\mathcal{S}(\mathbb{R})$ by $\|f\|_n^* = \|(N+1)^n f\|_2$. Further, define

$$\phi_n(x) = (2^n n!)^{-1/2}(-1)^n \pi^{-1/4} e^{x^2/2}\left(\frac{d}{dx}\right)^n e^{-x^2}$$

The functions $(\phi_n)_{n\in I_+}$ are called the **Hermite functions**.

Lemma 2. The set $(\phi_n)_{n\in I_+}$ is an orthonormal basis for $L_2(\mathbb{R})$.

Proof. Let $H_n(x) = (-1)^n e^{x^2}(\frac{d}{dx})^n e^{-x^2}$ and let $w(x,t) = e^{2xt-t^2}$. Then,

$$\left(\frac{d}{dt}\right)^n w(x,t)\Big|_{t=0} = \left(\frac{d}{dt}\right)^n \left(e^{x^2} e^{-(x-t)^2}\right)\Big|_{t=0} = (-1)^n e^{x^2}\left(\frac{d}{du}\right)^n e^{-u^2}\Big|_{u=x} = H_n(x)$$

So we have that

$$w(x,t) = e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n \tag{1}$$

Further,

$$\frac{d}{dt}\left(w(x,t)\right) = (2x-2t)w(x,t)$$

So, by substituting (1), we have

$$\begin{aligned} 0 &= \frac{d}{dt}\left(\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n\right) - 2x \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n + 2t \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n \\ &= \sum_{n=1}^{\infty} \frac{H_n(x)}{(n-1)!} t^{n-1} - 2x \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n + 2 \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^{n+1} \\ &= \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n - 2x \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n + 2 \sum_{n=1}^{\infty} \frac{H_{n-1}(x)}{(n-1)!} t^n \end{aligned}$$

Then, by equating coefficients, we have

$$\frac{H_{n+1}(x)}{n!} - \frac{2xH_n(x)}{n!} + \frac{2H_{n-1}(x)}{(n-1)!} = 0$$

for all $n \geq 1$ and hence,

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0 \tag{2}$$

Similarly,

$$\frac{d}{dx}\left(w(x,t)\right) = 2tw(x,t)$$

So, by substituting (1), we have

$$0 = \sum_{n=0}^{\infty} \frac{H_n'(x)}{n!} t^n - 2 \sum_{n=1}^{\infty} \frac{H_{n-1}(x)}{(n-1)!} t^n$$

Then, by equating coefficients we get

$$H_n'(x) = 2nH_{n-1}(x) \tag{3}$$

Now, substituting (3) into (2) we have

$$H_{n+1}(x) - 2xH_n(x) + H_n'(x) = 0$$

$$\Rightarrow H_{n+1}'(x) - 2H_n(x) - 2xH_n'(x) + H_n''(x) = 0$$

And, therefore,

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0 \tag{4}$$

Let $u_n(x) = e^{-x^2/2} H_n(x)$. Then,

$$\begin{aligned} u_n''(x) &= -u_n(x) + x^2 u_n(x) - 2xe^{-x^2/2} H_n'(x) + e^{-x^2/2} H_n''(x) \\ &= (x^2-1)u_n(x) + e^{-x^2/2}\left(H_n''(x) - 2xH_n'(x)\right) \\ &= (x^2-1)u_n(x) + e^{-x^2/2}\left(-2nH_n(x)\right) && \text{by (4)} \\ &= (x^2-2n-1)u_n(x) \end{aligned}$$

So, we have that

$$u_n''(x) + (2n+1-x^2)u_n(x) = 0 \tag{5}$$

Now, let $n, m \in I_+$ such that $n \neq m$. Then, by (5),

$$u_m(x)u_n''(x) + (2n+1)u_m(x)u_n(x) = 0 \quad \text{and} \quad u_n(x)u_m''(x) + (2m+1)u_n(x)u_m(x) = 0 \tag{6}$$

Then, by (6),

$$\begin{aligned} \frac{d}{dx}\left(u_n'(x)u_m(x) - u_m'(x)u_n(x)\right) + 2(n-m)u_m(x)u_n(x) &= u_n''(x)u_m(x) - u_m''(x)u_n(x) + 2(n-m)u_m(x)u_n(x) \\ &= (-2n-1+2m+1+2n-2m)u_n(x)u_m(x) \\ &= 0 \end{aligned}$$

So,

$$2(n-m)u_m(x)u_n(x) = -\frac{d}{dx}\left(u_n'(x)u_m(x) - u_m' u_n(x)\right)$$

And therefore,

$$2(n-m)\int_{\mathbb{R}} u_m(x)u_n(x)dx = -(u_n'(x)u_m(x) - u_m'(x)u_n(x)|_{-\infty}^{\infty} = 0$$

Since $u_n'(x)u_m(x) - u_m'(x)u_n(x) = p(x)e^{-x^2/2} \to 0$ as $|x| \to \infty$ where p is a polynomial of degree $n+m+1$. Therefore u_m and u_n are orthogonal. For $n = m$, first substitute $n-1$ for$n$ in (2) and then multiply through by $H_n(x)$ to get

$$H_n^2(x) - 2xH_n(x)H_{n-1}(x) + 2(n-1)H_n(x)H_{n-2}(x) = 0 \tag{7}$$

for $n \geq 2$. Similarly,

$$H_{n-1}(x)H_{n+1}(x) - 2xH_{n-1}(x)H_n(x) + 2nH_{n-1}^2(x) = 0 \tag{8}$$

Then, subtracting (8) from (7) we have

$$0 = H_n^2(x) + 2(n-1)H_n(x)H_{n-2}(x) - H_{n-1}(x)H_{n+1}(x) - 2nH_{n-1}^2(x)$$

Therefore, by multiplying through by $e^{-x^2/2}$ and integrating, we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \left(u_n^2(x) + 2(n-1)u_n(x)u_{n-2}(x) - u_{n-1}(x)u_{n+1}(x) - 2nu_{n-1}^2(x)\right) dx \\ &= \int_{\mathbb{R}} \left(u_N^2(x) - 2nu_{n-1}^2(x)\right) dx && \text{by orthogonality} \end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}} u_n^2(x)dx &= 2n \int_{\mathbb{R}} u_{n-1}^2(x)dx \\ &= 4n(n-1) \int_{\mathbb{R}} u_{n-2}^2(x)dx \\ &. \\ &. \\ &= 2^n n! \int_{\mathbb{R}} e^{-x^2/2}dx \\ &= 2^n n! \sqrt{\pi}\end{aligned}$$

Therefore, $(\phi_n)_{n\in I_+}$ is orthonormal. To prove the Hermite functions are an orthonormal basis for $L_2(\mathbb{R})$ it suffices to prove that if

$$\int_{\mathbb{R}} e^{-x^2/2}H_n(x)f(x)dx = 0 \quad \text{for all } n \in I_+ \text{ then } f = 0$$

So, suppose $\int_{\mathbb{R}} e^{-x^2/2}H_n(x)f(x)dx = 0$ for all $n \in I_+$. Then, for any $t \in \mathbb{R}$,

$$\frac{\left(\frac{-it}{2}\right)^n}{n!} \int_{\mathbb{R}} e^{-x^2/2}H_n(x)f(x)dx = 0$$

and hence,

$$\begin{aligned}0 &= \sum_{n=0}^{\infty} \frac{\left(\frac{-it}{2}\right)^n}{n!} \int_{\mathbb{R}} e^{-x^2/2}H_n(x)f(x)dx \\ &= \int_{\mathbb{R}} \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} \left(\frac{-it}{2}\right)^n e^{-x^2/2}f(x)dx \\ &= \int_{\mathbb{R}} e^{-tx+t^2/4}e^{-x^2/2}f(x)dx\end{aligned}$$

Therefore,

$$\mathcal{F}\left(e^{-x^2/2}f\right)(t) = \int_{\mathbb{R}} e^{-x^2/2}f(x)e^{-tx}dx = 0$$

And since the Fourier Transform is an isometry on $L_2(\mathbb{R})$ we have that $e^{-x^2/2}f(x) = 0$ and therefore $f = 0$. □

Lemma 3. The family of seminorms $(\|\cdot\|_n^*)_{n\in I_+}$ is a directed family which is equivalent to the $(\|\cdot\|_{n,m,2})_{n,m\in I_+}$ family of seminorms on $\mathcal{S}(\mathbb{R})$.

Proof. Let A^o denote A or A^*. Our first goal is to prove the inequality

$$\|A^o_{(1)}A^o_{(2)}\cdots A^o_{(m)}f\|_2 \leq \|(N+m)^{m/2}f\|_2$$

Let $c_n = (-1)^n(2^n n!\sqrt{\pi})^{-1/2}$ so that $\phi_n(x) = c_n e^{x^2/2}(\frac{d}{dx})^n e^{-x^2/2}$. Then

$$A\phi_n(x) = \frac{c_n}{\sqrt{2}}\left(2xe^{x^2/2}\left(\frac{d}{dx}\right)^n\left[e^{-x^2/2}\right] + e^{x^2/2}\left(\frac{d}{dx}\right)^n\left[-2xe^{-x^2/2}\right]\right)$$

Further,

$$\left(\frac{d}{dx}\right)^n\left[-2xe^{-x^2/2}\right] = -2\sum_{\nu=0}^{n}\binom{n}{\nu}(x)^{(\nu)}(e^{-x^2})^{(n-\nu)} = -2\left(x(e^{-x^2})^{(n)} + n(e^{-x^2})^{(n-1)}\right)$$

So, we have that,

$$A\phi_n(x) = \frac{c_n}{\sqrt{2}}\left(-2ne^{-x^2}\left(\frac{d}{dx}\right)^{n-1}\left[e^{-x^2}\right]\right) = \sqrt{n}\phi_{n-1}(x)$$

Also,

$$\begin{aligned} A^*\phi_n(x) &= \frac{1}{\sqrt{2}}x\phi_n(x) - \frac{c_n}{\sqrt{2}}xe^{x^2/2}\left(\frac{d}{dx}\right)^n\left[e^{-x^2}\right] - \frac{c_n}{\sqrt{2}}e^{x^2/2}\left(\frac{d}{dx}\right)^{n+1}\left[e^{-x^2}\right] \\ &= -\frac{1}{\sqrt{2}}\frac{(-1)^n}{(2^n n!\sqrt{\pi})^{1/2}}e^{x^2/2}\left(\frac{d}{dx}\right)^{n+1}\left[e^{-x^2}\right] \\ &= \sqrt{n+1}\phi_{n+1}(x) \end{aligned}$$

Therefore, $N\phi_n(x) = A^*A\phi_n(x) = n\phi_n(x)$. Now, let $f \in \mathcal{S}(\mathbb{R})$. Then, by lemma 2, there exists $(a_n)_{n\in I_+}$ so that $f = \sum_{n=0}^{\infty} a_n\phi_n$. Then,

$$\begin{aligned} \|A^o_{(1)}A^o_{(2)}\cdots A^o_{(m)}f\|_2 &\le \left(\sum_{n=0}^{\infty}\left(\sqrt{n+1}\sqrt{n+2}\cdots\sqrt{n+m}\right)^2 a_n^2\right)^{1/2} \\ &\le \left(\sum_{n=0}^{\infty}(n+m)^m a_n^2\right)^{1/2} \\ &= \|(N+m)^{m/2}f\|_2 \end{aligned}$$

Now, let $n, m \in I_+$ and assume $n > m$. By our claim,

$$\begin{aligned} \|(N+1)^n f\|_2 + \|(N+1)^m f\|_2 &\le C\|(N+2n)^{2n}f\|_2 + C'\|(N+2m)^{2m}f\|_2 \\ &\le C''\left(\left(\sum_{k=0}^{\infty}(k+2n)^{2n}a_k^2\right)^{1/2} + \left(\sum_{k=0}^{\infty}(k+2m)^{2m}a_k^2\right)^{1/2}\right) \\ &\le 2C''\|(N+2n)^n f\|_2 \\ &\le C'''\|(N+1)^n f\|_2 \end{aligned}$$

So, $(\|\cdot\|_n^*)_{n\in I_+}$ is a directed family of seminorms. The fact that the seminorms $(\|\cdot\|_n^*)_{n\in I_+}$ are equivalent to the seminorms $(\|\cdot\|_{n,m,2})_{n,m\in I_+}$ follows immediately from our claim and the equation

$$xf = \frac{1}{\sqrt{2}}\left(A + A^*\right)f$$

and hence,

$$x^k\left(\frac{d}{dx}\right)^m f = \left(\frac{1}{\sqrt{2}}\right)^{k+m}\left(A + A^*\right)^k\left(A - A^*\right)^m f$$

□

We are now ready to prove the N-Representation Theorem for $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$.

Theorem 1 (The N-Representation Theorem for $\mathcal{S}(\mathbb{R})$). Let s be the set of sequences $(a_n)_{n\in I_+}$ in $\mathbb{C}$ with the property

$$\sup_{n\in I_+} |a_n|n^m < \infty \quad \text{for all } m \in I_+$$

Topologize s by defining the seminorms

$$\|(a_n)_{n\in I_+}\|_m^2 = \sum_{n=0}^{\infty}(n+1)^{2m}|a_n|^2$$

Let $f \in \mathcal{S}(\mathbb{R})$. Then the sequence $(a_n)_{n\in I_+}$, where $a_n = \int_{\mathbb{R}} f(x)\phi_n(x)dx$, is in s and the map $f \mapsto (a_n)_{n\in I_+}$ is a topological isomorphism.

Proof. Define $\psi : \mathcal{S}(\mathbb{R}) \to s$ by $\psi(f) = (a_n)_{n\in I_+}$ where $a_n = \int_{\mathbb{R}} f(x)\phi_n(x)dx$. Let $n \in I_+$. From the proof of lemma 2, we saw that $N\phi_n = n\phi_n$. Now, let $f \in \mathcal{S}(\mathbb{R})$. Since $(\phi_n)n \in I_+$ is an orthonormal basis for $L_2(\mathbb{R})$ there exists $(a_n)n \in I_+$ such that $f = \sum_{n=0}^{\infty} a_n\phi_n$. Well,

$$\sum_{n=0}^{\infty} a_n n^m \phi_n = \sum_{n=0}^{\infty} a_n N^m \phi_n = N^m f \in L_2(\mathbb{R})$$

So,

$$\sum_{n=0}^{\infty} |a_n|^2 n^{2m} = \|N^m f\|_2^2 < \infty$$

hence,

$$\sup_{n\in I_+} |a_n|n^m < \infty$$

and therefore $(a_n)_{n\in I_+} \in s$. Further,

$$\|f\|_m^* = \|(N+1)^m f\|_2 = \|\sum_{n=0}^{\infty} a_n(N+1)^m\phi_n\|_2 = \|\sum_{n=0}^{\infty} a_n(n+1)^m\phi_n\|_2$$

And,

$$\|\sum_{n=0}^{\infty} a_n(n+1)^m\phi_n\|_2 = \left(\sum_{n=0}^{\infty}(n+1)^2m|a_n|^2\right)^{1/2} = \|(a_n)_{n\in I_+}\|_m$$

Therefore, $\|f\|_m^* = \|(a_n)_{n\in I_+}\|_m$. Further, since $\|\cdot\|_m^*$ is actually a norm on $\mathcal{S}(\mathbb{R})$ we have that ψ is injective. Let $(a_n)_{n\in I_+}$. For $N \in \mathbb{N}$, let $f_N = \sum_{n=0}^{N} a_n\phi_n$. Then, if $N < M$,

$$\|f_N - f_m\|_m^* = \|(N+1)^m(f_N - f_M)\|_2 = \left(\sum_{n=N+1}^{M}(n+1)^{2m}|a_n|^2\right)^{1/2} \to 0$$

as $N, M \to \infty$. Therefore $(f_N)_{N\in I_+}$ is Cauchy in each $\|\cdot\|_m^*$ and thus Cauchy in each $\|\cdot\|_{n,m,2}$ by lemma 3 and hence Cauchy in each $\|\cdot\|_{n,m,\infty}$ by lemma 1, i.e., $(f_N)_{N\in I_+}$ is Cauchy in $\mathcal{S}(\mathbb{R})$. Therefore, there exists $f \in \mathcal{S}(\mathbb{R})$ so that $f_N \to f$ in $\mathcal{S}(\mathbb{R})$ and hence $f_N \to f$ in $L_2(\mathbb{R})$. Thus, $f = \sum_{n=0}^{\infty} a_n\phi_n$ and so ψ is onto. Lastly we want to show ψ is a homeomorphism. If $(f_n)_{n\in I_+} \subset \mathcal{S}(\mathbb{R})$ such that $f_n \to f$ with respect to $\|\cdot\|_m^*$ then

$$\|\psi(f_n) - \psi(f)\|_m = \|(a_n)_{n\in I_+} - (b_n)_{n\in I_+}\|_m = \|f_n - f\|_m^* \to 0$$

So ψ is continuous. Further, if $((a_n^m)_{n\in I_+})_{m\in I_+} \subset s$ such that $(a_n^m)_{n\in I_+} \to (b_n)_{n\in I_+}$ as $m\to\infty$ with respect to $\|\cdot\|_m$ then

$$\|\psi^{-1}((a_n)_{n\in I_+}) - \psi^{-1}((b_n)_{n\in I_+})\|_m^* = \|f^m - f\|_m^* = \|(a_n)_{n\in I_+} - (b_n)_{n\in I_+}\|_m \to 0$$

So ψ^{-1} is continuous and therefore ψ is a topological isomorphism.

□

Theorem 2 (The N-Representation Theorem for $\mathcal{S}'(\mathbb{R})$). Let $T \in \mathcal{S}'(\mathbb{R})$ and let $b_n = T(\phi_n)$ for all $n \in I_+$. Then for some $m \in I_+$, we have $|b_n| \le C(n+1)^m$ for all $n \in I_+$. Conversely, if $|b_n| \le C(n+1)^m$ for all $n \in I_+$, there is a unique $T \in \mathcal{S}'(\mathbb{R})$ with $T(\phi_n) = b_n$. Further, if $T \in \mathcal{S}'(\mathbb{R})$ and $b_n = T(\phi_n)$ then $\sum_{n=0}^{\infty} b_n\phi_n$ converges in the $\sigma(\mathcal{S}'(\mathbb{R}), \mathcal{S}(\mathbb{R}))$-topology to T.

Proof. Let $T \in \mathcal{S}'(\mathbb{R})$. First, we want to show there exists $m \in I_+$ and $C > 0$ such that $|T(\phi)| \le C\|\phi\|_m$ for all $\phi \in \mathcal{S}(\mathbb{R})$. Well, $T^{-1}(B(0,1))$ is open, where $B(0,1)$ is the ball centered at 0 of radius 1, so there exists an open neighborhood of zero $N \subseteq T^{-1}(B(0,1))$ such that $N \cap_{k=1}^n U_{m_k,x_k,\epsilon_k}$ where
$U_{m_k,x_k,\epsilon_k} = \{y \in \mathcal{S}(\mathbb{R}) : \|y - x_k\|_{m_k} < \epsilon_k\}$. Further, for all $k = 1, ..., n$ we have that $0 \in U_{m_k,x_k,\epsilon_k}$ so there exists $\delta_k > 0$ such that $U_{m_k,0,\delta_k} \subseteq U_{m_k,x_k,\epsilon_k}$. Then $M = \cap_{k=1}^n U_{m_k,0,\delta_k}$ is an open neighborhood of zero and $\phi \in M$ if and only if $\|\phi\|_{m_k} < \delta_k$ for all $k = 1, \ldots, n$. Also, note that $M \subseteq N \subseteq T^{-1}(B(,0,1))$. Since $(\|\cdot\|_n)_{n\in I_+}$ is directed, there exists $m \in I_+$ and $C > 0$ so that $\|\phi\|_{m_1} + \cdots + \|\phi\|_{m_n} \le C\|\phi\|_m$ for all $\phi \in \mathcal{S}(\mathbb{R})$. Let $\epsilon = \min\{\delta_1, \ldots, \delta_n\}$. Then, for $k = 1, \ldots, n$,

$$\left\|\frac{\epsilon\phi}{2C\|\phi\|_m}\right\|_{m_k} \le \left\|\frac{\epsilon\phi}{2C\|\phi\|_m}\right\|_{m_1} + \cdots + \left\|\frac{\epsilon\phi}{2C\|\phi\|_m}\right\|_{m_n} \le C\left\|\frac{\epsilon\phi}{2C\|\phi\|_m}\right\|_m = \frac{\epsilon}{2} < \epsilon_k$$

So, $\frac{\epsilon\phi}{2C\|\phi\|_m} \in M \subseteq T^{-1}(B(0,1))$ hence

$$\left|T\left(\frac{\epsilon\phi}{2C\|\phi\|_m}\right)\right| \le 1$$

and therefore

$$|T(\phi)| \le \frac{2C}{\epsilon}\|\phi\|_m$$

Hence,

$$|b_n| = |T(\phi_n)\| \le C\|\phi_n\|_m = C(n+1)^m$$

Conversely, suppose $(b_n)_{n\in I_+} \subset \mathbb{C}$ such that $|b_n| \leq C(n+1)^m$ for some $m \in I_+$. Let $(a_n)_{n\in I_+} \in s$. Define $B : s \to \mathbb{C}$ by $B((a_n)_{n\in I_+}) = \sum_{n=0}^{\infty} b_n a_n$. Then,

$$\begin{aligned}
|B((a_n)_{n\in I_+})| &\leq \sum_{n=0}^{\infty} |b_n||a_n| \\
&\leq \sum_{n=0}^{\infty} C(n+1)^m |a_n| \\
&\leq C\sum_{n=0}^{\infty} (n+1)^{-1}(n+1)^{m+1}|a_n| \\
&\leq \left(\sum_{n=0}^{\infty} (n+1)^{2m+2}|a_n|^2\right)^{1/2} \left(\sum_{n=0}^{\infty} \frac{1}{(n+1)^2}\right)^{1/2} \\
&= C\sqrt{\frac{\pi^2}{6}}\|(a_n)_{n\in I_+}\|_{m+1}
\end{aligned}$$

So B is a continuous linear functional on s. Then, if ψ is the topological isomorphism from $\mathcal{S}(\mathbb{R})$ into s, we have that $B \circ \psi \in \mathcal{S}'(\mathbb{R})$. Hence, if $T = B \circ \psi$ then

$$T\left(\sum_{n=0}^{\infty} a_n\phi_n\right) = B((a_n)_{n\in I_+}) = \sum_{n=0}^{\infty} a_n b_n$$

In particular, $T(\phi_n) = b_n$. Lastly, if $b_n = T(\phi_n)$ and $f \in \mathcal{S}(\mathbb{R})$. Then we can write $f = \sum_{n=0}^{\infty} a_n\phi_n$ and

$$\begin{aligned}
\left(\sum_{n=0}^{\infty} b_n\phi_n\right)(f) &= \int_{\mathbb{R}} \left(\sum_{n=0}^{\infty} b_n\phi_n(x)\right) f(x)dx \\
&= \sum_{n=0}^{\infty} b_n \int_{\mathbb{R}} \phi_n(x)f(x)dx \\
&= \sum_{n=0}^{\infty} T(\phi_n)a_n \\
&= T\left(\sum_{n=0}^{\infty} a_n\phi_n\right) \\
&= T(f)
\end{aligned}$$

Hence $\sum_{n=0}^{\infty} b_n\phi_n = T$ in the $\sigma(\mathcal{S}'(\mathbb{R}), \mathcal{S}(\mathbb{R}))$-topology. □

Theorem 3 (Regularity Theorem for Distributions). Let $T \in \mathcal{S}'(\mathbb{R})$. Then $T = D^n g$ for some polynomially bounded continuous function g and $n \in I_+$, that is,

$$T(f) = \int_{\mathbb{R}} (-1)^n g(x) \left(D^n f\right)(x)dx \quad \text{for all } f \in \mathcal{S}(\mathbb{R})$$

Proof. Let $T \in \mathcal{S}'(\mathbb{R})$. Let $b_n = T(\phi_n)$. Then, by the last theorem, $|b_n| \leq C(n+1)^m$ for some $m \in I_+$ and $C > 0$. Let $a_n = b_n(n+1)^{-(m+3)}$. Note that

$$\|\phi_n\|_\infty \leq \|\phi_n'\|_1 \leq c\|(1+x^2)\phi_n'\| \leq c'(n+1)^{3/2}$$

and so,

$$\begin{aligned}\sum_{n=0}^{\infty}|a_n|\|\phi\|_\infty &= \sum_{n=0}^{\infty}\frac{|b_n|}{(n+1)^{m+3}}\|\phi\|_\infty \\ &\leq \sum_{n=0}^{\infty}\frac{C}{(n+1)^3}\|\phi\|_\infty \\ &\leq \sum_{n=0}^{\infty}\frac{c'}{(n+1)^{3/2}} \\ &\leq \infty\end{aligned}$$

Therefore, $\sum_{n=0}^{\infty} a_n\phi_n$ converges uniformly to some continuous F on $\mathbb{R}$.

Then we have that

$$\begin{aligned}(N+1)^{m+3}F &= (N+1)^{m+3}\sum_{n=0}^{\infty}a_n\phi_n \\ &= (N+1)^{m+3}\sum_{n=0}^{\infty}\frac{b_n}{(n+1)^{m+3}}\phi_n \\ &= \sum_{n=0}^{\infty}\frac{b_n}{(n+1)^{m+3}}(N+1)^{m+3}\phi_n \\ &= \sum_{n=0}^{\infty}\frac{b_n}{(n+1)^{m+3}}(n+1)^{m+3}\phi_n \\ &= \sum_{n=0}^{\infty}b_n\phi_n \\ &= T\end{aligned}$$

where convergence is in the $\sigma(\mathcal{S}'(\mathbb{R}),\mathcal{S}(\mathbb{R}))$-topology by theorem 2. So, we have that $T = (N+1)^{m+3}F$. It remains to show that $T = D^n g$ for some polynomially bounded continuous function g which is fairly easy to convince ourselves of but quite tedious to prove formally. One simply has to do integration by parts many times.

□

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